Tackling Stackelberg Network Interdiction against a Boundedly Rational Adversary

Tien Mai[1] , **Avinandan Bose**[2] , **Arunesh Sinha**[3] , **Thanh Nguyen**[4] and **Ayushman Kumar Singh**[5]
[1]Singapore Management University
[2]University of Washington
[3]Rutgers University
[4]University of Oregon
[5]Indian Institute of Technology, Delhi
atmai@smu.edu.sg, avibose@cs.washington.edu, arunesh.sinha@rutgers.edu,
thanhhng@cs.uoregon.edu, ayushman.ksingh.iitdelhi@gmail.com

Abstract

This work studies Stackelberg network interdiction games — an important class of games in which a defender first allocates (randomized) defense resources to a set of critical nodes on a graph while an adversary chooses its path to attack these nodes accordingly. We consider a *boundedly rational* adversary in which the adversary's response model is based on a *dynamic* form of classic logit-based (quantal response) discrete choice models. The resulting optimization is non-convex and additionally, involves complex terms that sum over exponentially many paths. We tackle these computational challenges by presenting new efficient algorithms with solution guarantees. First, we present a near optimal solution method based on path sampling, piece-wise linear approximation and mixed-integer linear programming (MILP) reformulation. Second, we explore a dynamic programming based method, addressing the exponentially-many-path challenge. We then show that the gradient of the non-convex objective can also be computed in polynomial time, which allows us to use a gradient-based method to solve the problem efficiently. Experiments based on instances of different sizes demonstrate the efficiency of our approaches in achieving near-optimal solutions.

1 Introduction

Network interdiction is a well-studied topic in Artificial Intelligence. There are many practical problems [Smith and Song, 2020], such as in cyber systems, that can be modeled as a network interdiction problem. In literature, many variations in models of network interdiction exist, and consequentially, a variety of techniques have been used for solving different types of these problems. Our work focuses on a particular type in which there is a set of critical nodes to protect within a larger network. We employ a popular network interdiction model [Fulkerson and Harding, 1977; Israeli and Wood, 2002], where the interdictor (defender) uses a randomized allocation of limited defense resources for the critical nodes. The adversary traverses the graphs starting from an origin and reaching a destination. There is an interaction with the defender only if the adversary crosses any critical node. The interaction is modeled using a leader-follower (Stackelberg) game where the defender first randomly allocates resources and then the adversary chooses its path accordingly.

Motivated by the fact that human adversaries in real-world security domains often act non-optimally [Tambe, 2011], we model the adversary behavior in our game setting using a dynamic Quantal Response model (an instance of well-known dynamic discrete choice (DDC) models [Rust, 1987; Aguirregabiria and Mira, 2010]). While many real world security applications have benefited from bounded rational Quantal Response model in single shot game settings [Tambe, 2011; Yang *et al.*, 2012; Fang *et al.*, 2016; Bose *et al.*, 2022], to the best of our knowledge, existing works in sequential network interdiction unrealistically assume perfectly rational adversaries and make use of the linearity to utilize linear programming techniques to tackle the problem [Smith *et al.*, 2009; Smith and Song, 2020]. We are the first to explore the DDC model of bounded rational adversaries in the network interdiction setting and formulate the defender's problem as a nonlinear optimization, leading to the requirement of solving the network interdiction problems via nonlinear optimization techniques.

While there is a closed form of the DDC adversary choice probabilities in our game setting, which is mathematically interesting in itself, the closed form presents computational challenges as the naive computation of any such probability involves reasoning about exponentially many paths from origin to destination and is a non-convex problem. This presents challenges beyond those observed in the single shot setting with quantal responding adversary [Fang *et al.*, 2016; Mai and Sinha, 2022]. Thus, we address the challenge of solving such complex non-convex optimization problem for the defender with two different novel approximation algorithms.

First, we introduce an MILP-based method, named **LiSD** (**Li**nearization via **S**ampling and **D**iscretization). The solu-

tion of **LiSD** is a bounded approximation for the interdiction problem. **LiSD** is the result of an innovative combination of path-sampling with piece-wise linear approximation (PL) techniques. Path sampling tackles the computational challenge of exponentially many paths while PL provides a near-optimal defender strategy solution with a guaranteed bound.

Second, we propose an efficient **Dyn**amic **P**rogramming method, named **DynP**. Essentially, **DynP** provides a compact and tractable formulations of the defender utility function and the optimization objective's gradient even though these terms involve summing over exponentially many paths. This is accomplished by exploiting recursive relationships among adversary utility-related terms across different paths that involves in the defender's optimal strategy computation. By employing dynamic programming, we can follow a gradient descent approach that is computationally efficient at each step to optimize the defender strategy. Furthermore, while **DynP** is computationally efficient, it does not guarantee global optimality due to the non-convexity of the defender problem. We thus identify a special case in which the adversary can visit only one (any one) critical node and show that the optimization is unimodal in that case, implying that this problem can be solved optimally in a tractable manner using gradient descent. We further identify specific conditions under which the solution to the restricted problem provides approximation guarantees for the original unrestricted one.

Notation: Boldface characters represent matrices or vectors or sets, and a_i denotes the i-th element of $\mathbf{a}$ if $\mathbf{a}$ is indexable. We use $[m]$, to denote the set $\{1, \ldots, m\}$.

2 Related Work

Dynamic discrete choice (DDC) models. From the seminal work of [Rust, 1987], DDC models have been widely studied and used to analyze sequential looking-forward choice behaviors and have various applications, e.g., on fertility and child mortality [Wolpin, 1984], on job matching and occupational choice [Miller, 1984], on bus engine replacement [Rust, 1987], and on route choice analysis [Fosgerau *et al.*, 2013; Mai *et al.*, 2015]. Among existing DDC models, the logit-based DDC has been popular due to its closed-form formulation [Rust, 1987]. This model can be viewed as a dynamic version of the well-known multinomial logit (or Quantal Response) model [McFadden, 1981; Train, 2003]. In transportation modeling, logit-based DDC was utilized to develop models to predict people's boundedly rational path-choice behavior [Fosgerau *et al.*, 2013; Mai *et al.*, 2015]. As highlighted in [Zimmermann and Frejinger, 2020], such a model presents synergies with the *stochastic shortest path* problem [Bertsekas and Tsitsiklis, 1991].

Network interdiction. Our work is a boundedly rational version of the well-studied shortest path interdiction problem [Fulkerson and Harding, 1977; Israeli and Wood, 2002]. Existing work only consider perfectly rational adversaries [Smith *et al.*, 2009; Smith and Song, 2020]. The shortest path and other network interdiction problems with perfectly rational adversaries are generally NP-hard and have strong connections with the areas of bi-level optimization [Dempe *et al.*, 2015] and robust optimization [Ben-Tal and Nemirovski, 2002]. We refer the readers to [Smith and Song, 2020] for a comprehensive review. Our work explores the DDC framework to model bounded rational adversaries, resulting in a significantly more challenging defender problem as it involves complex nonlinear optimization. Besides, there are other variant models where the problem data is not perfectly known to players [Cormican *et al.*, 1998], or where the players repeatedly make their actions [Sefair and Smith, 2016], or where online learning is involved [Borrero *et al.*, 2016].

Network security games and others. Our work also relates to static Stackelberg security game models with Quantal Response adversaries [Yang *et al.*, 2011; Yang *et al.*, 2012; Haghtalab *et al.*, 2016; Mai and Sinha, 2022; Černỳ *et al.*, 2021; Milec *et al.*, 2020; Bose *et al.*, 2023b]. In dynamic models named as network security games [Jain *et al.*, 2011], the set-up is different from our work as in this work the *rational* adversary aims to reach a target and stop, whereas in our work the *boundedly rational* adversary can attack multiple targets. Other related works along this line only consider *zero-sum* network security game setting [Xue *et al.*, 2021; Xue *et al.*, 2022]. A Quantal Response type relaxation for network security game was also studied, where the focus in on smart predict and optimize [Wang *et al.*, 2020], however, the optimize part is done using standard non-linear solver such sequential quadratic program with no guarantees.

There are other related game models where players act in a graph-based environment, including pursuit-evasion and security patrol games [Zhang *et al.*, 2019; Basilico *et al.*, 2009; Basilico *et al.*, 2017]. However, these works do not consider the attacker's bounded rationality. Additionally, their strategy spaces and problem settings are characterized differently which involve aspects of real-time information or alarm signals., etc.

3 Problem Formulation

3.1 Stackelberg Network Interdiction Games

Our network interdiction problem is a leader-follower game with a single adversary. The game is played on a network (graph) $(\mathcal{S}, \mathcal{A})$ where $\mathcal{S}$ is a set of nodes $\mathcal{S} = \{1, 2, \ldots, |\mathcal{S}|\}$, and $\mathcal{A}$ is a set of arcs. We formulate the problem as a two-player network interdiction game . The follower (adversary) takes a path through this network, which is sampled from a distribution as described below. The origin $s_o \in \mathcal{S}$ is a given starting node. In our problem, we also assume the existence of a sink (or destination) node $s_d \in \mathcal{S}$ that the adversary ultimately reaches. Let $\mathcal{L}$ be the set of critical nodes (i.e., subset of nodes in the network) that the defender can interfere or alter. From the leader's (defender's) viewpoint, the aim is to assign M resources to nodes $s \in \mathcal{L}$; each such assignment is a defender pure strategy. Further, nodes and resources are of certain types such that nodes of a given type can only be protected by resources of that same kind. Let there be K types of nodes. Let the number of resources of each type k be M_k, hence $\sum_{k\in[K]} M_k = M$. Also, let $\{\mathcal{L}_k\}_{k\in[K]}$ be a partition of the set of nodes $\mathcal{L}$ by the types of the nodes.

A mixed strategy is a randomized allocation resulting in a coverage vector $\mathbf{x} = \{x_s,\ s \in \mathcal{L}, \sum_{s\in\mathcal{L}_k} x_s \leq M_k, \forall k \in [K]\}$ where x_s is the marginal probability of covering node s, which then impacts the adversary's path choice probabilities. Given a node $s \in \mathcal{S}$, if the adversary crosses this node, then the defender gets a node-specific reward $r^l(s, x_s)$. The *Stackelberg equilibrium* can be computed by solving the following problem [Yang *et al.*, 2012; Mai and Sinha, 2022]:

$$\max_{\mathbf{x}} \quad \mathcal{F}^l(\mathbf{x}) = \sum\nolimits_{\tau\in\Omega} R^l(\tau|\mathbf{x}) P^f(\tau|\mathbf{x}) \qquad \text{(OPT)}$$

$$\text{subject to} \quad \sum\nolimits_{s\in\mathcal{L}_k} x_s \leq M_k,\ \forall k \in [K] \qquad (1)$$

$$x_s \in [L^x, U^x],\ \forall s \in \mathcal{L},$$

where $R^l(\tau|\mathbf{x}) = \sum_{s\in\mathcal{L}\cap\tau} r^l(s, x_s)$ is the defender's accumulated reward on path τ and $P^f(\tau|\mathbf{x})$ is the probability the attacker follows the path τ (of which computation is discussed in the behavior modeling part). Here, $[L^x, U^x]$ represent the required lower bound and upper bound on the coverage probability for each node in the critical set $\mathcal{L}$.

3.2 Boundedly Rational Adversary Behavior

We model the adversary's bounded rational behavior using the dynamic discrete choice framework (and specifically the logit-based recursive path choice model [Fosgerau *et al.*, 2013]). A known property in this setting is that the bounded rational adversary chooses a policy that is equivalent to a static multinomial logit (MNL) discrete choice model over all possible paths [Fosgerau *et al.*, 2013].

Concretely, let $U(\tau|s_0, \mathbf{x}) = \sum_{s\in\tau} v(s; \mathbf{x})$ be the deterministic long-term utility of the adversary when starting in s_0; if $s_0 = s_o$, then we simply write $U(\tau|\mathbf{x})$. Here, $v(s; \mathbf{x})$ is the adversary's utility associated with node s when the defender's strategy is $\mathbf{x}$. Given $\mathbf{x}$, the probability the adversary follows a path τ can be computed as follows [Fosgerau *et al.*, 2013]:

$$P^f(\tau|\mathbf{x}) = \frac{e^{U(\tau;\mathbf{x})/\mu}}{Z}, \text{ where } Z = \sum\nolimits_{\tau\in\Omega} e^{U(\tau;\mathbf{x})/\mu}, \qquad (2)$$

given Ω is the set of all possible paths and μ is the parameter which governs the follower's rationality. Thus, we can view the logit-based dynamic discrete choice formulation as a soft version of the shortest weighted path problem from the source s_o to destination s_d. Given the adversary behavior model, the adversary's expected utility can be computed as an expectation over all paths, as follows:

$$\mathcal{E}^f(\mathbf{x}) = \sum\nolimits_{\tau\in\Omega} P^f(\tau|\mathbf{x}) U(\tau; \mathbf{x})$$

Our Prop. 1 shows that the adversary's expected utility approaches the best accumulated utility (smallest path weight) as μ tends to zero (we drop the fixed strategy $\mathbf{x}$ for simplicity).

Proposition 1. *Let $\tau^* = argmax_{\tau\in\Omega} U(\tau)$ (i.e., the best path for the adversary) and $L^* = |U(\tau^*)|$. Let $\Omega^* = \{\tau;\ U(\tau) = L^*\}$ and $\alpha = U(\tau^*) - \max_{\tau\in\Omega\setminus\Omega^*} U(\tau)$. We obtain:*

$$|\mathcal{E}^f - U(\tau^*)| \leq (L^* + 1)/(1 + {|\Omega^*|}/{|\Omega\setminus\Omega^*|} e^{\alpha/\mu}).$$

As a result, $\lim_{\mu\to 0} \mathcal{E}^f = U(\tau^*)$.[1]

[1] All proofs, if not presented, are included in the appendix.

4 Common Binary Search Framework

Overall, (OPT) is computationally challenging since the objective not only involves an exponential number of paths in the network but also is non-convex. To address this computational challenge, we propose two new different algorithms which share the common underlying binary search framework. The purpose is to reduce the original fractional (OPT) to a simpler non-fractional problem. These algorithms then differ in applying different efficient techniques to solve each binary search step. We elaborate them in subsequent sections.

Essentially, we write the objective of (OPT) as follows:

$$\mathcal{F}^l(\mathbf{x}) = \frac{\sum_{\tau\in\Omega} R^l(\tau|\mathbf{x}) \exp\left({U(\tau;\mathbf{x})}/{\mu}\right)}{\sum_{\tau\in\Omega} \exp\left({U(\tau;\mathbf{x})}/{\mu}\right)}$$

$\mathcal{F}^l(\mathbf{x})$ has a fractional non-convex form. A typical way to simplify this structure is to use the Dinkelbach transform and a binary search algorithm [Dinkelbach, 1967] to convert the original problem into a sequence of simpler ones. We use binary search to write (OPT) equivalently as: $\max_\lambda \left\{\lambda \;\middle|\; \exists \mathbf{x} \text{ s.t. } \mathcal{F}^l(\mathbf{x}) \geq \lambda\right\}$ which is equivalent to finding a maximum value of $\lambda \in \mathbb{R}$ such that the following subproblem:

$$\max_{\mathbf{x}} \Big\{ \sum_{\tau\in\Omega} R^l(\tau|\mathbf{x}) \exp\big(\frac{U(\tau;\mathbf{x})}{\mu}\big) - \lambda \sum_{\tau\in\Omega} \exp\big(\frac{U(\tau;\mathbf{x})}{\mu}\big) \Big\} \quad (3)$$

has a non-negative optimal objective value. Overall, (3) is still non-convex, but no longer fractional. In addition, the set Ω of all feasible paths can be huge and may not be enumerable. Therefore, we propose two different algorithms (as elaborated next) to tackle these challenges in solving (3).

5 Linearization via Sampling and Discretizing

We describe our first near-optimal method, **LiSD**, which involves exploring path-sampling with piece-wise linear approximation (PL) techniques to approximate (3) by a MILP. Path sampling tackles the computational challenge of exponentially many paths while PL provides a near-optimal defender strategy solution with a guaranteed bound for (3).

5.1 Sample Average Approximation

We first approximate the sum over Ω via sample average approximation. That is, we select a feasible solution $\mathbf{x}_0$ to create a fixed distribution over paths in Ω. By dividing the objective of (3) by $\sum_{\tau\in\Omega} \exp({U(\tau;\mathbf{x}_0)}/{\mu})$, which is a constant, we aim to maximize the following objective function:

$$G(\mathbf{x}, \lambda) = \mathbb{E}_{\tau\sim\mathcal{D}(\mathbf{x}_0)}\big[R^l(\tau|\mathbf{x}) \exp\big(\widetilde{U}(\tau|\mathbf{x})\big) - \lambda \exp\big(\widetilde{U}(\tau|\mathbf{x})\big)\big] \quad (4)$$

where $\widetilde{U}(\tau|\mathbf{x}) = \frac{U(\tau;\mathbf{x})}{\mu} - \frac{U(\tau;\mathbf{x}_0)}{\mu}$, and $\mathcal{D}(\mathbf{x}_0)$ is the distribution over paths τ with probabilities $P^f(\tau|\mathbf{x}_0)$ (Eq. 2).

We now can approximate the objective function $g(\mathbf{x}, \lambda)$ by sample average approximation. Specifically, let $\tau_1, \ldots, \tau_N$ be N samples from $\mathcal{D}(\mathbf{x}_0)$, we approximate $g(\mathbf{x}, \lambda)$ by:

$$\widehat{G}^N(\mathbf{x}, \lambda) = \frac{1}{N} \sum_{n\in[N]} \Big[R^l(\tau_n|\mathbf{x}) e^{\widetilde{U}(\tau_n|\mathbf{x})} - \lambda e^{\widetilde{U}(\tau_n|\mathbf{x})}\Big] \qquad (5)$$

Essentially, the approximation $\widehat{G}^N(\mathbf{x},\lambda)$ converges to $G(\mathbf{x},\lambda)$ almost surely as $N \to \infty$ and the approximation errors can be bounded as shown in Proposition 2.

Proposition 2. *For any given $\xi > 0$, we have:*

$$\mathbb{P}\Big(\big|\widehat{G}^N(\boldsymbol{x},\lambda) - G(\boldsymbol{x},\lambda)\big| \geq \xi\Big) \leq 2\exp\left(-\frac{2N\xi^2}{\mathcal{M}^2}\right)$$

where $\mathcal{M} = \max_{\tau,\boldsymbol{x}} \big\{J(\tau,\boldsymbol{x})\big\} - \min_{\tau,\boldsymbol{x}} \big\{J(\tau,\boldsymbol{x})\big\}$

$$J(\tau,\boldsymbol{x}) = R^l(\tau|\boldsymbol{x})\exp\big(\widetilde{U}(\tau|\boldsymbol{x})\big) - \exp\big(\widetilde{U}(\tau|\boldsymbol{x})\big)$$

Proposition 2 implies that $\widehat{G}^N(\mathbf{x},\lambda)$ will converge to the true function $G(\mathbf{x},\lambda)$ in probability with an exponential rate as the number of samples N increases. This is a direct result from Hoeffding's inequality [Hoeffding, 1994].

5.2 Piece-wise Linear (PL) Approximation

We now further approximate $\widehat{G}^N(\mathbf{x},\lambda)$ by a PL function, allowing the subproblem to be solved to near-optimality via a MILP solver. First, for each τ_n, we introduce new variables $u_n = R^l(\tau_n|\mathbf{x})$ and $v_n = \frac{\widetilde{U}(\tau_n|\mathbf{x})}{\mu}$. We now can re-write the objective function, $\widehat{G}^N(\mathbf{x},\lambda)$ accordingly, as follows:

$$\widehat{G}^N(\mathbf{x},\lambda) = \frac{1}{N}\sum\nolimits_{n\in[N]}\Big(u_n\exp(v_n) - \lambda\exp(v_n)\Big)$$

Let L_n and U_n be an lower and upper bounds of v_n. The PL approximation can be done by partitioning each interval $[L_n, U_n]$ into K sub-intervals of equal size, and introducing K binary variables $z_n^1,\ldots,z_n^n$ such that $z_n^1 \geq z_n^2 \geq \ldots \geq z_n^K$, to represent each interval. Intuitively, $z_n^k = 1$ implies the k^{th} sub-interval involves in the approximation of $\exp(v_n)$ and $z_n^k = 0$, otherwise. Let $\Delta_n = {}^{(U_n - L_n)}/_K$ (i.e., the size of each interval) and δ_n^k, $k \in [K]$ is the slop of function e^{v_n} in the interval $[L_n + \Delta_n(k-1), L_n + \Delta_n k]$:

$$\delta_n^k = \frac{\exp(L_n + \Delta_n k) - \exp(L_n + \Delta_n(k-1))}{\Delta_n}$$

Each component $\exp(v_n)$ can be approximated as follows:

$$\exp(v_n) \approx \exp(L_n) + \Delta_n \sum\nolimits_{k\in[K]} \delta_n^k z_n^k$$

We then can re-write the sub-problem (3) as follows:

$$\max_{\mathbf{x},\mathbf{z},\mathbf{u},\mathbf{v}} \frac{1}{N}\sum\nolimits_{n\in[N]}(u_n - \lambda)\Big(\exp(L_n) + \Delta_n \sum\nolimits_{k\in[K]}\delta_n^k z_n^k\Big) \tag{MINLP}$$

$$\text{s.t. } z_n^k \geq z_n^{k+1};\ k \in [K-1], n \in [N] \tag{6}$$

$$u_n = R^l(\tau_n|\mathbf{x}) \text{ and } v_n = {}^{\widetilde{U}(\tau_n|\mathbf{x})}/_{\mu} \tag{7}$$

$$v_n = L_n + \Delta_n \sum\nolimits_{k\in[K]} z_n^k + \kappa_n \tag{8}$$

$$\mathbf{x} \in \mathcal{X}, \mathbf{z}_n \in \{0,1\}^K, \kappa_n \in [0, \Delta_n] \tag{9}$$

which maximizes the piece-wise approximation of $\widehat{G}^N(\mathbf{x},\lambda)$. The additional variable κ_n captures the gap between v_n and the binary approximation $L_n + \Delta_n \sum_{k\in[K]} z_n^k$.

Finally, there are only some bi-linear terms left to be linearized in the objective function. We do that using McCormick inequalities. Specifically, let L_n^u and U_n^u be lower and upper bounds of u_n, we introduce new variables s_n^k to present $(u_n - \lambda)z_n^k$, we can now linearize the bi-linear term $(u_n - \lambda)z_n^k$ with the following additional constraints:

$$s_n^k \leq (U_n^u - \lambda)z_n^k;\ s_n^k \geq (L_n^u - \lambda)z_n^k \tag{10}$$

$$s_n^k \leq (u_n - \lambda) - (L_n^u - \lambda)(1 - z_n^k) \tag{11}$$

$$s_n^k \geq (u_n - \lambda) - (U_n^u - \lambda)(1 - z_n^k) \tag{12}$$

The above three constraints guarantee that when $z_n^k = 1$, then $s_n^k = u_n - \lambda$. Conversely, when $z_n^k = 0$, then $s_n^k = 0$.

By combining the above new variable s_n^k and constraints with (MINLP), we obtain the MILP reformulation:

$$\max_{\mathbf{x},\mathbf{z},\mathbf{u},\mathbf{v},\mathbf{s}} \frac{1}{N}\sum\nolimits_{n\in[N]}\Big((u_n - \lambda)e^{L_n} + \Delta_n \sum\nolimits_{k\in[K]}\delta_n^k s_n^k\Big) \tag{MILP}$$

s.t. Constraints (6–12) are satisfied.

We further establish a performance bound for PL approximation. We first remark, from the definition of $\widehat{G}^N(\mathbf{x},\lambda)$, that:

$$\begin{cases} \widehat{G}^N(\mathbf{x},\lambda) \geq 0 & \text{if } \lambda \leq \min_{n,\mathbf{x}} R^l(\tau_n|\mathbf{x}) = \min_n\{L_n^u\} \\ \widehat{G}^N(\mathbf{x},\lambda) \leq 0 & \text{otherwise.} \end{cases}$$

So, it is sufficient to consider $\lambda \in [\min_n\{L_n^u\}, \max_n\{U_n^u\}]$. This allows us to state Proposition 3 below.

Proposition 3. *Assume that $\lambda \in \big[\min_n\{L_n^u\}, \max_n\{U_n^u\}\big]$, let $\widehat{\boldsymbol{x}}^{NK}$ be an optimal solution to* (MILP) *and $\boldsymbol{x}^*$ be optimal for average approximation sub-problem $\max_{\boldsymbol{x}} \widehat{G}^N(\boldsymbol{x},\lambda)$, then we obtain the following inequality:*

$$\Big|\widehat{G}^N(\widehat{\boldsymbol{x}}^{NK},\lambda) - \widehat{G}^N(\boldsymbol{x}^*,\lambda)\Big| \leq \frac{2BN}{K}$$

where $B = \big(\max_n\{U_n^u\} - \min_n\{L_n^u\}\big)\max_n\big\{e^{U_n}(U_n - L_n)\big\}$.

From an intuitive standpoint, augmenting K would diminish the approximation error of the PL approximation. Conversely, augmenting N has a dual effect: while it lessens the error arising from path sampling, it simultaneously heightens the cumulative error stemming from all the samples. In fact, to drive the bound closer to zero, Proposition 3 indicates that it's necessary that the rate of increase for K should surpass that of N. We further investigate this dual effect by looking at the quality of a solution returned from (MILP) w.r.t the original sub-problem $\max_{\mathbf{x}} G(\mathbf{x},\lambda)$. A performance bound is provided in Theorem 1, which implies that, under the condition $N \leq \frac{\xi K}{6B}$, the solution given by the PL approximation will converge in probability to a true optimal solution, with an exponential rate.

Theorem 1. *Assume that $\lambda \in \big[\min_n\{L_n^u\}, \max_n\{U_n^u\}\big]$. Let $\widehat{\boldsymbol{x}}^{NK}$ be an optimal solution to* (MILP) *and $\boldsymbol{x}^*$ be optimal for $\max_{\boldsymbol{x}} G(\boldsymbol{x},\lambda)$, then given any $\xi > 0$, if we choose N,K such that $\frac{N}{K} \leq \frac{\xi}{6B}$, then we have:*

$$\mathbb{P}(|G(\widehat{\boldsymbol{x}}^{NK},\lambda) - G(\boldsymbol{x}^*,\lambda)| \geq \xi) \leq 4e^{-\frac{2N\xi^2}{9\mathcal{M}^2}}.$$

This result can be employed to establish (theoretical) estimates for N and K to achieve a desired performance.

Corollary 1. *For any given $\alpha, \beta > 0$, $\beta \in (0,1)$, if we choose $N \geq \ln\left(\frac{4}{\beta}\right)\frac{9\mathcal{M}^2}{2\alpha^2}$ and $K \geq \frac{6NB}{\alpha}$, then $\left|G(\widehat{\boldsymbol{x}}^{NK},\lambda) - G(\boldsymbol{x}^*,\lambda)\right| \leq \alpha$ occurs with probability $1-\beta$.*

The above estimates might shed light on how N, K depends on the performance criteria α, β. We note that these estimates would be conservative, as in practice we may need much smaller N, K to achieve the desired performance. A final note is that one can employ an off-the-shelf solver (e.g. CPLEX or GUROBI) to solve (MILP). Although this program would be large in size, SOTA solvers can efficiently handle very large MILPs, aided by powerful machines.

6 Dynamic Programming Based Solution

The above MILP approximation involves binary variables and would be intractable in large scenarios. We thus propose an alternative new algorithm, ***DynP*** that also follows binary search, but at each binary step, (i) it presents a non-trivial compact representation of the objective function based on the creation of a *dynamic program,* which handles an exponential number of paths; and (ii) it applies a gradient ascent-based method to efficiently solve the resulting compact problem.

6.1 Compact Representation

We can rearrange terms in the objective of sub-problem (3) according to critical nodes as follows:

$$g(\mathbf{x},\lambda) = \sum\nolimits_{s\in\mathcal{L}} \sum_{\substack{\tau\in\Omega\\ \tau\ni s}} r^l(s,x_s)\exp\left({}^{U(\tau;\mathbf{x})}/_{\mu}\right) - \lambda\Big[\sum\nolimits_{\tau'\in\Omega}\exp\left({}^{U(\tau';\mathbf{x})}/_{\mu}\right)\Big] \quad (13)$$

Since $g(\mathbf{x},\lambda)$ is differentiable, this maximization problem can be solved for a local maximum by a gradient-based method. One of the key challenges is the computation of $g(\mathbf{x},\lambda)$, which, if done naively, would require enumerating exponentially many paths on Ω. We next show that $g(\mathbf{x},\lambda)$ has a compact form, which allows us to compute $g(\mathbf{x},\lambda)$ and its gradient efficiently via dynamic programming.

For a compact representation of $g(\mathbf{x},\lambda)$, we introduce the following new terms for all nodes $s, s' \in \mathcal{S}$:

$$Z_s = \sum_{\tau\in\Omega^{s_d}(s)} \exp\left({}^{U(\tau;\mathbf{x})}/_{\mu}\right) \text{ and } Y^s_{s'} = \sum_{\tau\in\Omega(s',s)} \exp\left({}^{U(\tau;\mathbf{x})}/_{\mu}\right)$$

where $\Omega^{s_d}(s)$ is the set of all paths from s to the destination s_d and $\Omega(s',s)$ is the set of all paths from s' to s.

The objective $g(\mathbf{x},\lambda)$ can be now re-formulated as follows:

$$g(\mathbf{x},\lambda) = \sum\nolimits_{s\in\mathcal{L}} r^l(s,x_s)Y^s_{s_o}Z_s - \lambda Z_{s_o}, \quad (14)$$

where s_o is the origin. Although these new terms still involve exponentially many paths in $\Omega^{s_d}(s)$ and $\Omega(s',s)$, they can be computed efficiently via dynamic programming.

Indeed, $\{Z_s\}_s$ can be computed recursively as follows:

$$Z_s = \begin{cases} \sum_{s'\in N(s)} \exp\left({}^{v(s;\mathbf{x})}/_{\mu}\right) Z_{s'} & \text{if } s \neq s_d \\ 1 & \text{if } s = s_d, \end{cases}$$

Algorithm 1: Dynamic **P**rogramming based algorithm (**DynP**) to solve *Maximizing* $g(\boldsymbol{x},\lambda)$

Input: $\lambda \in \mathbb{R}$ and an initial value of $\mathbf{x}$
while *not converged* **do**
  Given $\mathbf{x}$, solve the system $\mathbf{H} = (\mathbf{I}-\mathbf{M})^{-1}\mathbf{B}$ and $\mathbf{J}^{\mathbf{H},j} = (\mathbf{I}-\mathbf{M})^{-1}\mathbf{J}^{\mathbf{M},j}\mathbf{H}$ for all j
  Compute $g(\mathbf{x},\lambda)$ and $\frac{\partial g(\mathbf{x},\lambda)}{\partial x_s}$ using Eq. 14, 15.
  Update $\mathbf{x}$ using a projected gradient method
end

where $N(s) = \{s' \in \mathcal{S} |\ (s,s') \in \mathcal{A}\}$, is the set of possible next nodes that can be reached in one hop from node $s \in \mathcal{S}$.

Let $\mathbf{M}$ be a matrix of size $|\mathcal{S}\times\mathcal{S}|$ with entries defined as:

$$M_{ss'} = \exp\left({}^{v(s|\mathbf{x})}/_{\mu}\right)\ \forall\ s\in\mathcal{S}, s'\in N(s)$$

Then $\mathbf{Z} = \{Z_s, s\in\mathcal{S}\}$ is a solution to the linear system $\mathbf{Z} = \mathbf{MZ} + \mathbf{b}$, where $\mathbf{b}$ is of size $|\mathcal{S}| \times 1$ with zero entries except $b_{s_d} = 1$. Similarly, we can compute $\mathbf{Y}^s = \{Y^s_{s'}\}_{s'}$ recursively:

$$Y^s_{s'} = \begin{cases} \sum_{s''\in N(s')} \left(\exp\left({}^{v(s';\mathbf{x})}/_{\mu}\right)\right) Y_{s''} & \text{if } s' \neq s \\ 1 & \text{if } s' = s. \end{cases}$$

Clearly, $\mathbf{Y}^s$ is a solution to the linear system $\mathbf{Y}^s = \mathbf{MY}^s + \mathbf{b}^s$, where $\mathbf{b}^s$ is of size $|\mathcal{S}|$ with zeros everywhere except $b^s_s = 1$.

Since $\mathbf{Y}^s$ and $\mathbf{Z}$ are solutions to the systems $\mathbf{Y}^s = \mathbf{MY}^s + \mathbf{b}^s$ and $\mathbf{Z} = \mathbf{MZ} + \mathbf{b}$, respectively, $\forall s \in \mathcal{S}$, the objective $g(\mathbf{x},\lambda)$ can be computed via solving $|\mathcal{L}|+1$ system of linear equations. Finally, we see that all the above linear systems rely on the common matrix $\mathbf{M}$. We can group them all into only one linear system. Let $\mathbf{H}$ be a matrix of size $(|\mathcal{S}|) \times (|\mathcal{L}|+1)$ in which the 1st to $|\mathcal{L}|$-th columns are vectors $\mathbf{Y}^s$, $s \in \mathcal{L}$ and the last column is $\mathbf{Z}$. Let $\mathbf{B}$ be a matrix of size $(|\mathcal{S}|)\times(|\mathcal{L}|+1)$ in which the 1st to $|\mathcal{L}|$-th columns are vectors $\mathbf{b}^s$, $s \in \mathcal{L}$ and the last column is $\mathbf{b}$. We see that $\mathbf{H}$ is a solution to the linear system $(\mathbf{I}-\mathbf{M})\mathbf{H} = \mathbf{B}$. Thus, in general, we can solve only one linear system to obtain all $\mathbf{Y}^s$ and $\mathbf{Z}$. This way should be scalable when the size of $\mathcal{L}$ increases.

6.2 Gradient Computation

We aim at employing the gradient-based approach to solve the binary search step: $\max_{\mathbf{x}}\{g(\mathbf{x},\lambda)\}$ (aka. Eq. 3). The core is to compute the gradient $\left\{{}^{\partial g(\mathbf{x},\lambda)}/_{\partial x_s}\right\}$. According to Eq. 14, this gradient computation requires differentiating through the matrices $\mathbf{Z}$ and $\{\mathbf{Y}^s\}$ (or equivalently, differentiating through the matrix $\mathbf{H}$). We first present our Proposition 4:

Proposition 4. *($\boldsymbol{I-M}$) is invertible in a cycle-free network.*

Prop. 4 allows us to compute the matrix $\mathbf{H}$ as: $\mathbf{H} = (\mathbf{I}-\mathbf{M})^{-1}\mathbf{B}$. By taking the derivatives of both sides w.r.t x_j, $j \in \mathcal{L}$, we obtain the following: for all $j \in \mathcal{L}$,

$$\mathbf{J}^{\mathbf{H},j} = (\mathbf{I}-\mathbf{M})^{-1}\mathbf{J}^{\mathbf{M},j}(\mathbf{I}-\mathbf{M})^{-1}\mathbf{B} = (\mathbf{I}-\mathbf{M})^{-1}\mathbf{J}^{\mathbf{M},j}\mathbf{H},$$

where $\mathbf{J}^{\mathbf{H},j}$ and $\mathbf{J}^{\mathbf{M},j}$ are the gradient matrices of $\mathbf{H}$ and $\mathbf{M}$ w.r.t x_j, i.e., $\mathbf{J}^{\mathbf{H},j}$ is a matrix of size $|\mathcal{S}|\times(|\mathcal{L}|+1)$ with entries $\mathbf{J}^{\mathbf{H},j}_{ss'} = {}^{\partial H_{ss'}}/_{\partial x_j}$, and $\mathbf{J}^{\mathbf{M},j}$ is a matrix of size $(|\mathcal{S}| \times |\mathcal{S}|)$

with entries $\mathbf{J}^{\mathbf{M},j}_{ss'} = {}^{\partial M_{ss'}}/_{\partial x_j}$, for any $j \in \mathcal{L}$. Let $\mathbf{R}^l(\mathbf{x})$ be a matrix of size $1 \times |\mathcal{L}|$ with entries $r^l(s, x_s)$ for $s \in \mathcal{L}$. We use $A_{S,T}$ to denotes a sub-matrix of A which uses the rows in set S and columns in set T. If S or T is a singleton, e.g., $S = \{s_o\}$ or $T = |\mathcal{L}| + 1$, then we write it as s_o or $|\mathcal{L}| + 1$.

As a result, we now can compute the required gradient as follows for all $s \in \mathcal{L}$ where $\circ$ denotes Hadamard product:

$$\frac{\partial g(\mathbf{x}, \lambda)}{\partial x_s} = \left(\mathbf{R}^l(\mathbf{x}) \circ \mathbf{J}^{\mathbf{H},s}_{s_o,\mathcal{L}} + \mathbf{J}^{R,s} \circ \mathbf{H}_{s_o,\mathcal{L}}\right) \times \mathbf{H}_{\mathcal{L},|\mathcal{L}|+1} + \left(\mathbf{R}^l(\mathbf{x}) \circ \mathbf{H}_{s_o,\mathcal{L}}\right) \times \mathbf{J}^{\mathbf{H},j}_{\mathcal{L},|\mathcal{L}|+1} - \lambda \mathbf{J}^{\mathbf{H}}_{s_o,|\mathcal{L}|+1} \tag{15}$$

We summarize the main steps to optimize $g(\mathbf{x}, \lambda)$ in Alg. 1.

Remark 1. *Alg. 1 only guarantees a local optimum due to the non-convexity of $g(\boldsymbol{x}, \lambda)$. The complexity is determined by the matrix inversion which, in worst case, is in $O(|\mathcal{S}|^3)$. The gradient descent loop runs $O({}^1/_\epsilon)$ to provide an additive ϵ approximation. Thus, the total complexity is $O\left(({}^1/_\epsilon)|\mathcal{S}||\mathcal{A}|\right)$. In practice, the gradients can be found via auto differentiation techniques, providing significantly more speed-up.*

6.3 A Natural Special Case

Separation of critical resources and/or privileges is an important concept in security [Lin *et al.*, 2023]. Following this principle, we analyze a special yet natural security design scenario where that the critical nodes $\mathcal{L}$ are well separated. Specifically, we assume that the cost of travelling between nodes in $\mathcal{L}$ is high. More formally, given a critical node $s \in \mathcal{L}$, let $\Delta^+(s)$ be the set of paths that cross s and at least another critical node in $\mathcal{L}$. Let $\beta_1, \beta_2 > 0$ such that:

$$\beta_1 = \max_{\mathbf{x}} \max_{s \in \mathcal{L}} \left\{ \frac{\sum_{\tau \in \Delta^+(s)} \exp\left({}^{U(\tau;\mathbf{x})}/_\mu\right)}{\sum_{\tau \in \Delta(s)} \exp\left({}^{U(\tau;\mathbf{x})}/_\mu\right)} \right\}$$
$$\beta_2 = \max_{\mathbf{x}} \left\{ \frac{\sum_{\tau \in \bigcup_s \{\Delta^+(s)\}} \exp\left({}^{U(\tau;\mathbf{x})}/_\mu\right)}{\sum_{\tau \in \bigcup_s \{\Delta(s)\}} \exp\left({}^{U(\tau;\mathbf{x})}/_\mu\right)} \right\}, \tag{16}$$

Intuitively, β_1 and β_2 are expected to be small if the cost of traveling between any two critical nodes in $\mathcal{L}$ is large. That is, $\beta_1, \beta_2 \to 0$ as $\sum_{\tau \in \Omega(s,s')} \exp\left({}^{U(\tau;\mathbf{x})}/_\mu\right) \to 0$, where $\Omega(s, s')$ consists of all paths from s to s', for any $s, s' \in \mathcal{L}$. Surprisingly, even though **DynP** only finds a locally optimal solution for (OPT) due to its non-convexity, we show that assuming small β_1 and β_2 provides approximation guarantees for the globally optimal solution value.

For this approximation, we need mild assumptions that the utilities have a linear form: $v(s; \mathbf{x}) = w^f_s x_s + t^f_s$ and $r^l(s; \mathbf{x}) = r^l(s, x_s) = w^l_s x_s + t^l_s$ for some constants $w^f_s, t^f_s, w^l_s, t^l_s$. We assume that $w^f_s < 0$ and $w^l_s > 0$, i.e., more resources x_s at s will lower adversary's utilities, and increase the defender's utility. This setting is intuitive for security settings [Yang *et al.*, 2012; Mai and Sinha, 2022].

We first introduce a restricted interdiction problem that can be solved *optimally* in a tractable manner using our efficient gradient descent-based method. We then present an important theoretical result showing how the restricted problem's solution yields an approximate solution of with the original problem for well separated critical nodes.

Let $\Delta(s)$ be the set of paths that cross a critical node s and do not cross any other node in $\mathcal{L}$. We consider the following restricted interdiction problem:

$$\max_{\mathbf{x}} \widetilde{\mathcal{F}}(\mathbf{x}) = \frac{\sum_{s \in \mathcal{L}, \tau \in \Delta(s)} r^l(s, x_s) \exp\left({}^{U(\tau;\mathbf{x})}/_\mu\right)}{\sum_{s \in \mathcal{L}, \tau \in \Delta(s)} \exp\left({}^{U(\tau;\mathbf{x})}/_\mu\right)} \tag{Approx-OPT}$$
$$\text{s.t.} \sum\nolimits_{s \in \mathcal{L}_k} x_s \le M_k, \ \forall k \in [K]$$
$$x_s \in [L^x, U^x], \ \forall s \in \mathcal{L}.$$

Intuitively, in this restricted problem (Approx-OPT), the adversary's path choices are restricted to a subspace of paths in the network which only cross a *single* critical node in $\mathcal{L}$. We denote by $\mathcal{X}$, the feasible set of the defender's interdiction strategies $\mathbf{x}$ that satisfy the constraints in (Approx-OPT).

Solution Relation with Original Problem (OPT)

We now theoretically analyze (Approx-OPT)'s solution in relation to our original problem (OPT). We prove that:

Theorem 2. *Let $\boldsymbol{x}^*$ be an approx. solution to* (Approx-OPT)*: $\max_{\boldsymbol{x} \in \mathcal{X}} \widetilde{\mathcal{F}}(\boldsymbol{x})$ such that $\widetilde{\mathcal{F}}(\boldsymbol{x}^*) \ge (1 - \epsilon) \max_{\boldsymbol{x}} \widetilde{\mathcal{F}}(\boldsymbol{x})$ for given $\epsilon > 0$, let $\kappa = \max_{\boldsymbol{x} \in \mathcal{X}} \sum_{s \in \mathcal{L}} |r^l(s, x_s)|$ be the maximal absolute reward that the defender can possibly achieve at a critical node, then we obtain:*

$$\mathcal{F}^l(\boldsymbol{x}^*) \ge \frac{(1 - \epsilon) \max_{\boldsymbol{x}} \left\{\mathcal{F}^l(\boldsymbol{x})\right\}}{(1 + \beta_1)(1 + \beta_2)} - \kappa \frac{\epsilon + \beta_1 + \beta_2 + \beta_1 \beta_2}{(1 + \beta_1)(1 + \beta_2)}.$$

Additionally, if $\boldsymbol{x}^*$ is an approx. solution with an additive error $\epsilon > 0$, we obtain the following bound:

$$\mathcal{F}^l(\boldsymbol{x}^*) \ge \left({}^1/_\eta\right) \max_{\boldsymbol{x}} \{\mathcal{F}^l(\boldsymbol{x})\} - \kappa\left({}^{(\eta-1)}/_\eta\right),$$

where $\eta = (1 + \beta_1)(1 + \beta_2)\left(1 + \frac{\epsilon}{\kappa + \min_{\boldsymbol{x} \in \mathcal{X}} \widetilde{\mathcal{F}}(\boldsymbol{x})}\right)$.

Note that $\max_{\mathbf{x} \in \mathcal{X}} \{\mathcal{F}^l(\mathbf{x})\}$ is the original problem (OPT) to find an optimal defender strategy. As stated previously, when the cost of traveling between any two critical nodes is high, (β_1, β_2) is close to zero, meaning the RHS of both inequalities in Theorem 2 will closely approximate the optimal solution value of (OPT).

Solving the Restricted Problem: To solve the restricted problem, we also apply binary search. It can be demonstrated that each sub-problem of the binary search can be effectively solved to optimality (or near-optimality) using a gradient-based method. Due to limited space, the details of this approach is provided in the appendix along with the sub-results that lead to the main Theorem 2.

7 Numerical Experiments

To illustrate the efficacy of our proposed algorithms, we perform experiments on synthetic data.

7.1 Experimental Settings

Data generation. We generate random graphs (cycle-free) with $|\mathcal{S}|$ vertices and edge probability p. We randomly choose $|\mathcal{L}|$ vertices (except source and destination) as the critical nodes that can be attacked. We set $|\mathcal{L}| = 0.8 \times |\mathcal{S}|$. In

addition, the defender weights $\{(w_j^l, t_j^l) \mid j \in [|\mathcal{L}|]\}$ are generated uniformly at random from the interval $[0, 1]$ and the adversary weights $\{(w_j^f, t_j^f) \mid j \in [|\mathcal{S}|]\}$ are generated at random from the interval $[-1, 0]$. Moreover, we used $p = 0.8, \mu = 2$.

Baseline. We approximate the sums over exponentially many paths in Equation 3 by sampling paths from the network and run gradient descent on this expression to estimate the optimal decision variable. To sample paths for the baseline, a resource allocation $\mathbf{x}$ is assigned to $\mathcal{L}$ and the follower is initially placed at the origin s_o. Its next node s_1 is sampled from the distribution

$$\pi^f(s|s_0, \mathbf{x}) = \frac{\exp\left(v(s;\mathbf{x})/\mu\right) Z_s}{\sum_{s' \in N(s_0)} \exp\left(v(s';\mathbf{x})/\mu\right) Z_{s'}},$$

where $s \in \{$*nodes having an edge to* $s_0\}$ and $N(s_0)$ is the set of outgoing nodes from s_0. Similarly $s_2, \ldots$ etc. are sampled till the destination s_d is reached. This sampling is repeated 1000 times per iteration and then average is taken to get the objective. Based on the gradients, the resource allocation $\mathbf{x}$ is updated which changes the transition probabilities and the process is repeated again until convergence. Ten different values of $\mathbf{x}$ were taken and the seed with the lowest loss was reported. We will compare this baseline against our near optimal MILP-based algorithm, **LiSD** and our dynamic programming based algorithm, **DynP**. To ensure fairness, all algorithms were run with the same number of epochs.

Choosing N and K for the LiSD To justify our choices of N and K for the **LiSP** described, we first fix $N = 90$ and vary K from 5 to 100. For each value of K, we generate 10 independent instances and solve them using the MILP approach. We then observe that the optimal values given by $K = 20$ are only 0.3% different from those given by the largest value of K (i.e., $K = 100$). Moreover, the optimal values given by $N = 90$ are only about 3% different from those from the largest value of N, i.e., $N = 150$. The numerical details can be found in the appendix. We therefore choose $N = 90$ and $K = 20$ for the **LiSD** approach. According to the above analyses, these choices would suffice to guarantee low practical approximation errors stemming from both path-sampling and PL approximation. We use GUROBI (a SOTA MILP solver) to solve (MILP). All our experiments were run on a 2.1 GHz CPU with 128GB RAM.

7.2 Numerical Comparison

We vary the number of nodes from 20 to 100. For each choice of number of nodes, we generate 20 independent instances and solve them by the three methods (Baseline, **DynP**, and **LiSD**). The rates of giving best objective values (over 20 instances) are reported in Table 1. **LiSD** consistently outperforms the other methods, by a large margin, in terms of the number of times it returns the best objective values. **DynP** performs better than Baseline for large-size settings, but worse than Baseline for small-size ones. Note that, among the four methods, only **LiSD** can guarantee near-optimal solutions. The computing times are not directly comparable, as GUROBI ran on several cores while the other gradient-based methods use only one CPU core. We however observe that, for instances of 100 nodes, the average computing times for the Baseline, **DynP**, and **LiSD** are approximately 3, 15 and 1.8 minutes.

# nodes	Baseline	(Ours) LiSD	(Ours) DynP
20	30%	**50%**	0%
40	20%	**65%**	15%
60	15%	**55%**	30%
80	20%	**45%**	35%
100	15%	**45%**	15%

Table 1: Rates of giving best objective values. Each measurement is computed using 20 independent instances.

# nodes	Baseline	(Ours) LiSD	(Ours) DynP
20	253.1%	**344.2%**	301.9%
40	54.5%	**63.2%**	55.9%
60	51.7%	**57.6%**	55.8%
80	25.3%	**34.7%**	30.4%
100	88.2%	**93.0%**	89.0%

Table 2: Average percentage improvements w.r.t the lowest objective values given by the four methods.

We further compare the objective values returned by the four methods by computing the percentage improvement of each objective w.r.t. the lowest objective value given by the four approaches. Specifically, we solve each instance to obtain 4 objective values. We then compute the percentage improvement of each objective value w.r.t the lowest value among the four values. The *average* percentage values are reported in Table 2 below, which show that **LiSD** performs the best, and **DynP** outperforms the Baseline.

8 Conclusion

Network interdiction game problems present a set of challenges that appear intractable to start with. In this work, we address some of these challenges by providing novel methods that efficiently solve a class of network interdiction problems with approximation guarantees. We are the first to study the dynamic Quantal Response model in the type of network interdiction studied in [Fulkerson and Harding, 1977; Israeli and Wood, 2002]. We believe this modeling and methodology contribution provides suggestions for other future research directions, such as a variant where the adversary's objective is to maximize a flow through the network or a setting where the leader would also need to make dynamic and time-dependent decisions. This online nature of the problem suggests possible future work in online learning problems such as [Bose *et al.*, 2023a]. It is interesting to analyse scenarios where network structures arise naturally such as ride pool matching [Bose and Varakantham, 2021] and are under threat from adversaries such as competing providers.

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9 Missing Proofs

9.1 Proof of Proposition 1.

Proof. We re-write the adversary's expected utility as follows:

$$\begin{aligned}
\mathcal{E}^f &= \sum_{\tau\in\Omega} U(\tau)\frac{\exp\left(\frac{U(\tau)}{\mu}\right)}{\sum_{\tau'}\exp\left(\frac{U(\tau')}{\mu}\right)} \\
&= \sum_{\tau\in\Omega^*} U(\tau)\frac{\exp\left(\frac{U(\tau)}{\mu}\right)}{\sum\limits_{\tau'\in\Omega^*}\exp\left(\frac{U(\tau')}{\mu}\right)+\sum\limits_{\tau'\in\Omega\setminus\Omega^*}\exp\left(\frac{U(\tau')}{\mu}\right)} \\
&+ \sum_{\tau\in\Omega\setminus\Omega^*} U(\tau)\frac{\exp\left(\frac{U(\tau)}{\mu}\right)}{\sum\limits_{\tau'\in\Omega^*}\exp\left(\frac{U(\tau')}{\mu}\right)+\sum\limits_{\tau'\in\Omega\setminus\Omega^*}\exp\left(\frac{U(\tau')}{\mu}\right)} \\
&= \frac{U(\tau^*)|\Omega^*|}{|\Omega^*|+\sum\limits_{\tau'\in\Omega\setminus\Omega^*}\exp\left(\frac{U(\tau')-U(\tau^*)}{\mu}\right)} + \sum_{\tau\in\Omega\setminus\Omega^*}\frac{U(\tau)\exp\left(\frac{U(\tau)-U(\tau^*)}{\mu}\right)}{|\Omega^*|+\sum\limits_{\tau'\in\Omega\setminus\Omega^*}\exp\left(\frac{U(\tau')-U(\tau^*)}{\mu}\right)} \qquad (17)
\end{aligned}$$

Let us denote $\mathcal{T} = \sum_{\tau'\in\Omega\setminus\Omega^*}\exp\left(\frac{U(\tau')-U(\tau^*)}{\mu}\right)$ for notational simplicity. We see that:

$$0 \le \mathcal{T} \le |\Omega\setminus\Omega^*|\exp\left(-\frac{\alpha}{\mu}\right) \qquad (18)$$

From (17), we have:

$$\begin{aligned}
|\mathcal{E}^f - U(\tau^*)| &\le \frac{\mathcal{T}}{|\Omega^*|+\mathcal{T}} + \left|\sum_{\tau\in\Omega\setminus\Omega^*}\frac{U(\tau)\exp\left(\frac{U(\tau)-U(\tau^*)}{\mu}\right)}{|\Omega^*|+\mathcal{T}}\right| \\
&\le \frac{\mathcal{T}}{|\Omega^*|+\mathcal{T}} + L^*\left|\frac{\mathcal{T}}{|\Omega^*|+\mathcal{T}}\right| \\
&= (L^*+1)\frac{1}{1+\frac{|\Omega^*|}{\mathcal{T}}} \overset{(i)}{\le} \frac{L^*+1}{1+\frac{|\Omega^*|}{|\Omega\setminus\Omega^*|}\exp\left(\frac{\alpha}{\mu}\right)},
\end{aligned}$$

where (i) is due to (18). We obtain the desired inequality. The limit $\lim_{\mu\to 0}\mathcal{E}^f = U(\tau^*)$ is just a direct result of this inequality, concluding our proof. □

9.2 Proof of Proposition 3

Proof. Let $\widehat{G}^{NK}(\mathbf{x})$ be the objective of (MINLP) as a function of $\mathbf{x}$ ($\mathbf{x}$ is fixed in the optimization problem). Since we approximate function $\exp(v_n)$ by a piece-wise linear function $\exp(v_n) \approx \widehat{T}(\mathbf{z}_n) = \exp(L_n) + \Delta_n\sum_{k\in[K]}\delta_n^k z_n^k$, according to the mean value theorem, the approximation error between $\exp(v_n)$ and its approximation can be bounded as

$$|\exp(v_n) - \widehat{T}(\mathbf{z}_n)| \le \max_{v\in[L_n,U_n]}\{\exp(v)\Delta_n\} \le \exp(U_n)\frac{U_n-L_n}{K}$$

This implies that, for any $\mathbf{x}$ we can bound the gap $|\widehat{G}^{NK}(\mathbf{x}) - \widehat{G}^{K}(\mathbf{x})|$ as

$$\begin{aligned}
|\widehat{G}^{NK}(\mathbf{x}) - \widehat{G}^{K}(\mathbf{x})| &\le \sum_n \max\{(u_n-\lambda)\}\exp(U_n)\frac{U_n-L_n}{K} \\
&\le \sum_n (\max_n\{U_n^u\} - \min_n\{U_n^u\})\exp(U_n)\frac{U_n-L_n}{K} \le \frac{NB}{K} \qquad (19)
\end{aligned}$$

Now, let $\widehat{\mathbf{x}}^{NK}$ be optimal for $\max_{\mathbf{x}}\{\widehat{G}^{NK}(\mathbf{x})\}$ and $\mathbf{x}^*$ is optimal for $\max_{\mathbf{x}}\{\widehat{G}^{N}(\mathbf{x})\}$. We have the following chain

$$\begin{aligned}
|\widehat{G}^{N}(\mathbf{x}^*) - \widehat{G}^{N}(\widehat{\mathbf{x}}^{NK})| &\le |\widehat{G}^{N}(\mathbf{x}^*) - \widehat{G}^{NK}(\widehat{\mathbf{x}}^{NK})| + |\widehat{G}^{NK}(\widehat{\mathbf{x}}^{NK}) - \widehat{G}^{N}(\widehat{\mathbf{x}}^{NK})| \\
&\overset{(a)}{\le} \frac{NB}{K} + |\widehat{G}^{N}(\mathbf{x}^*) - \widehat{G}^{NK}(\widehat{\mathbf{x}}^{NK})| \qquad (20)
\end{aligned}$$

where (a) is due to (19). We now consider two cases:

(i) If $\widehat{G}^N(\mathbf{x}^*) \geq \widehat{G}^{NK}(\widehat{\mathbf{x}}^{NK})$, then

$$\begin{aligned}|\widehat{G}^N(\mathbf{x}^*) - \widehat{G}^{NK}(\widehat{\mathbf{x}}^{NK})| &= \widehat{G}^N(\mathbf{x}^*) - \widehat{G}^{NK}(\widehat{\mathbf{x}}^{NK}) \\ &\leq \widehat{G}^N(\mathbf{x}^*) - \widehat{G}^{NK}(\mathbf{x}^*) \leq \frac{NB}{K}\end{aligned}$$

(ii) If $\widehat{G}^N(\mathbf{x}^*) \leq \widehat{G}^{NK}(\widehat{\mathbf{x}}^{NK})$, then

$$\begin{aligned}|\widehat{G}^N(\mathbf{x}^*) - \widehat{G}^{NK}(\widehat{\mathbf{x}}^{NK})| &= -\widehat{G}^N(\mathbf{x}^*) + \widehat{G}^{NK}(\widehat{\mathbf{x}}^{NK}) \\ &\leq -\widehat{G}^N(\widehat{\mathbf{x}}^{NK}) - \widehat{G}^{NK}(\widehat{\mathbf{x}}^{NK}) \leq \frac{NB}{K}\end{aligned}$$

Combine the two cases we get $|\widehat{G}^N(\mathbf{x}^*) - \widehat{G}^{NK}(\widehat{\mathbf{x}}^{NK})| \leq \frac{B}{K}$. Together with (20), we get the desired inequality:

$$|\widehat{G}^N(\mathbf{x}^*) - \widehat{G}^N(\widehat{\mathbf{x}}^{NK})| \leq \frac{2BN}{K}.$$

□

9.3 Proof of Theorem 1

Proof. Let $\widehat{\mathbf{x}}^N$ be optimal for $\max_{\mathbf{x}} \widehat{G}^N(\mathbf{x})$. We first write

$$\begin{aligned}\Big|G(\mathbf{x}^*,\lambda) - G(\widehat{\mathbf{x}}^{NK},\lambda)\Big| \leq &\Big|G(\mathbf{x}^*,\lambda) - \widehat{G}^N(\widehat{\mathbf{x}}^N,\lambda)\Big| \\ &+ \Big|\widehat{G}^N(\widehat{\mathbf{x}}^N,\lambda) - \widehat{G}^N(\widehat{\mathbf{x}}^{NK},\lambda)\Big| \\ &+ \Big|\widehat{G}^N(\widehat{\mathbf{x}}^{NK},\lambda) - G(\widehat{\mathbf{x}}^{NK},\lambda)\Big| \qquad (21)\end{aligned}$$

Therefore, for any $\xi \geq 0$, we obtain:

$$\begin{aligned}\mathbb{P}\Big(\Big|G(\mathbf{x}^*,\lambda) - G(\widehat{\mathbf{x}}^{NK},\lambda)\Big| \geq \xi\Big) \leq &\mathbb{P}\left(\Big|G(\mathbf{x}^*,\lambda) - \widehat{G}^N(\widehat{\mathbf{x}}^N,\lambda)\Big| \geq \frac{\xi}{3}\right) \\ &+ \mathbb{P}\left(\Big|\widehat{G}^N(\widehat{\mathbf{x}}^N,\lambda) - \widehat{G}^N(\widehat{\mathbf{x}}^{NK},\lambda)\Big| \geq \frac{\xi}{3}\right) \\ &+ \mathbb{P}\left(\Big|\widehat{G}^N(\widehat{\mathbf{x}}^{NK},\lambda) - G(\widehat{\mathbf{x}}^{NK},\lambda)\Big| \geq \frac{\xi}{3}\right) \qquad (22)\end{aligned}$$

Now, if we choose N, K such that $\frac{2BN}{K} \leq \frac{\xi}{3}$, then according to Proposition 3 we have:

$$\mathbb{P}\left(\Big|\widehat{G}^N(\widehat{\mathbf{x}}^N,\lambda) - \widehat{G}^N(\widehat{\mathbf{x}}^{NK},\lambda)\Big| \geq \frac{\xi}{3}\right) = 0 \qquad (23)$$

On the other hand, from Proposition (2) we can bound the probabilities as follows:

$$\begin{aligned}\mathbb{P}\left(\Big|G(\mathbf{x}^*,\lambda) - G(\widehat{\mathbf{x}}^{NK},\lambda)\Big| \geq \frac{\xi}{3}\right) &\leq 2\exp\left(-\frac{2N\xi^2}{9\mathcal{M}^2}\right) \\ \mathbb{P}\left(\Big|\widehat{G}^N(\widehat{\mathbf{x}}^{NK},\lambda) - G(\widehat{\mathbf{x}}^{NK},\lambda)\Big| \geq \frac{\xi}{3}\right) &\leq 2\exp\left(-\frac{2N\xi^2}{9\mathcal{M}^2}\right)\end{aligned}$$

Combine the above with (23), we obtain the desired bound for $\mathbb{P}\left(\Big|G(\mathbf{x}^*,\lambda) - G(\widehat{\mathbf{x}}^{NK},\lambda)\Big| \geq \xi\right)$, which concludes our proof. □

9.4 Proof of Corollary 1

This is a direct result of Theorem 1. That is, we replace ξ in Theorem 1 by α to get that if we choose $\alpha \geq \frac{6NB}{K}$ then

$$\mathbb{P}\left(\Big|G(\widehat{\mathbf{x}}^{NK},\lambda) - G(\mathbf{x}^*,\lambda)\Big| \leq \alpha\right) \geq 1 - 4\exp\Big(-\frac{2N\xi^2}{9\mathcal{M}^2}\Big)$$

To achieve $\mathbb{P}\left(\Big|G(\widehat{\mathbf{x}}^{NK},\lambda) - G(\mathbf{x}^*,\lambda)\Big| \leq \alpha\right) \geq 1 - \beta$, we need to choose N such that

$$\beta \geq 4\exp\Big(-\frac{2N\xi^2}{9\mathcal{M}^2}\Big)$$

implying

$$N \geq \ln\left(\frac{4}{\beta}\right)\frac{9\mathcal{M}^2}{2\alpha^2},$$

as desired.

9.5 Proof of Proposition 4

For any $n \in \mathbb{N}$, let us consider $\mathbf{M}^n = \underbrace{\mathbf{M} \times \mathbf{M} \times \ldots \mathbf{M}}_{n \text{ times}}$ with entries

$$\mathbf{M}^n_{ss'} = \sum_{\substack{(s_0,s_1,\ldots,s_n)\in\mathcal{S}^{n+1} \\ s_0=s, s_n=s'}} \left(\prod_{i=0}^{n-1} M_{s_i s_{i+1}} \right)$$

Recall that $M_{ss'} = \exp\left(\frac{v(s)}{\mu}\right)$ if $s' \in N(s)$ and $M_{ss'} = 0$ otherwise. We see that if $n > |\mathcal{S}|$, then for any sequence $(s_0, s_1, \ldots, s_n)$ there is at least a pair $s_j = s_k$, $0 \leq j, k \leq n$. Since the network is cycle-free, there is at least a pair (s_j, s_{j+1}) such that $M_{s_j s_{j+1}} = 0$, leading to the fact that $\mathbf{M}^n_{ss'} = \sum_{\substack{(s_0,s_1,\ldots,s_n)\in\mathcal{S}^{n+1} \\ s_0=s, s_n=s'}} \left(\prod_{i=0}^{n-1} M_{s_i s_{i+1}} \right) = 0$ for any $s, s' \in \mathcal{S}$. Thus, if $n > |\mathcal{S}|$ we have $\mathbf{M}^n = \mathbf{0}$. We select $n > |\mathcal{S}|$ and write

$$(\mathbf{I} - \mathbf{M}) \left(\sum_{t=0}^{n-1} \mathbf{M}^t \right) = (\mathbf{I} - \mathbf{M}^n) = \mathbf{I},$$

which implies $\det(\mathbf{I} - \mathbf{M}) \neq 0$, or equivalently, $\mathbf{I} - \mathbf{M}$ is invertible as desired.

9.6 Proofs of Theorems 2.

The proof of these theorem are based on two important lemmas, as explained below. Lemma 1 only applies when all the defender's rewards $r^l(s, x_s)$ are non-negative. We then handle the general case in Lemma 2. Intuitively, these two lemmas show relations in terms of the defender's utilities (aka. objective functions $F^l(\mathbf{x})$ and $\widetilde{F}(\mathbf{x})$) between the original problem (OPT) and the restricted problem (Approx-OPT) for any given defender's interdiction strategy $\mathbf{x}$.

Lemma 1. *If $r^l(s, x_s) \geq 0$ for any $\boldsymbol{x} \in \mathcal{X}$, then for any $\boldsymbol{x}$,*

$$\frac{1}{1+\beta_2} \widetilde{\mathcal{F}}(\boldsymbol{x}) \leq \mathcal{F}^l(\boldsymbol{x}) \leq (1+\beta_1) \widetilde{F}(\boldsymbol{x}).$$

The case that $r^l(s, x_s)$ would take negative values is more challenging to handle. The following lemma gives general inequalities for such a situation.

Lemma 2. *If we choose $\kappa = \sum_{s\in\mathcal{L}} \max_{\boldsymbol{x}} \left| r^l(s, x_s) \right|$, then for any $\boldsymbol{x} \in \mathcal{X}$,*

$$\frac{1}{1+\beta_2} \left(\widetilde{\mathcal{F}}(\boldsymbol{x}) + \kappa \right) \leq \mathcal{F}^l(\boldsymbol{x}) + \kappa \leq (1+\beta_1) \left(\widetilde{\mathcal{F}}(\boldsymbol{x}) + \kappa \right).$$

The proofs of the two lemmas are provided in the next sections. We will now use this two lemmas to prove Theorem 2 as follows.

According to Lemma 2, we obtain:

$$\frac{1}{1+\beta_2} \max_{\mathbf{x}} \left(\widetilde{\mathcal{F}}(\mathbf{x}) + \kappa \right) \leq \max_{\mathbf{x}} \left(\mathcal{F}^l(\mathbf{x}) + \kappa \right) \leq (1+\beta_1) \max_{\mathbf{x}} \left(\widetilde{\mathcal{F}}(\mathbf{x}) + \kappa \right), \tag{24}$$

leading to the following chain of inequalities:

$$\frac{1}{1+\beta_2} \left(\widetilde{\mathcal{F}}(\mathbf{x}^*) + \kappa \right) \leq \left(\mathcal{F}^l(\mathbf{x}^*) + \kappa \right) \leq \max_{\mathbf{x}} (\mathcal{F}^l(\mathbf{x}) + \kappa) \leq (1+\beta_1) \left(\widetilde{\mathcal{F}}(\mathbf{x}^*) + \kappa \right).$$

where $\mathbf{x}^*$ is the optimal defense strategy solution to (Approx-OPT). Since $\widetilde{\mathcal{F}}(\mathbf{x}^*) + \kappa, \mathcal{F}^l(\mathbf{x}^*) + \kappa$, and $\max_{\mathbf{x}}(\mathcal{F}^l(\mathbf{x}) + \kappa)$ are all positive, we have:

$$\frac{\mathcal{F}^l(\mathbf{x}^*) + \kappa}{\max_{\mathbf{x}}(\mathcal{F}^l(\mathbf{x}) + \kappa)} \geq \frac{\frac{1}{1+\beta_2} \left(\widetilde{\mathcal{F}}(\mathbf{x}^*) + \kappa \right)}{(1+\beta_1) \left(\widetilde{\mathcal{F}}(\mathbf{x}^*) + \kappa \right)} = \frac{1}{(1+\beta_1)(1+\beta_2)}$$

$$\text{which implies: } \mathcal{F}^l(\mathbf{x}^*) + \kappa \geq \frac{\max_{\mathbf{x}\in\mathcal{X}}\{\mathcal{F}^l(\mathbf{x})\}}{(1+\beta_1)(1+\beta_2)} + \frac{\kappa}{(1+\beta_1)(1+\beta_2)}$$

$$\text{or equivalently, } \mathcal{F}^l(\mathbf{x}^*) \geq \frac{\max_{\mathbf{x}\in\mathcal{X}}\{\mathcal{F}^l(\mathbf{x})\}}{(1+\beta_1)(1+\beta_2)} - \kappa \frac{\beta_1 + \beta_2 + \beta_1\beta_2}{(1+\beta_1)(1+\beta_2)}$$

which validate the first part of the theorem. For the second part, let $\mathbf{x}^*$ be the interdiction strategy solution such that $\widetilde{\mathcal{F}}(\mathbf{x}^*) \geq (1-\epsilon)\max_{\mathbf{x}} \widetilde{\mathcal{F}}(\mathbf{x})$, then by using (24), we obtain the following chain of inequalities:

$$\begin{aligned}\frac{1}{1+\beta_2}\left(\widetilde{\mathcal{F}}(\mathbf{x}^*)+\kappa\right) &\leq \frac{1}{1+\beta_2}\max_{\mathbf{x}}\left(\widetilde{\mathcal{F}}(\mathbf{x})+\kappa\right) \leq \max_{\mathbf{x}}\left(\mathcal{F}^l(\mathbf{x})+\kappa\right) \\ &\leq (1+\beta_1)\max_{\mathbf{x}}\left(\widetilde{\mathcal{F}}(\mathbf{x})+\kappa\right) \leq (1+\beta_1)\left(\frac{1}{1-\epsilon}\widetilde{\mathcal{F}}(\mathbf{x}^*)+\kappa\right) \\ &\leq \frac{1+\beta_1}{1-\epsilon}(\widetilde{\mathcal{F}}(\mathbf{x}^*)+\kappa)\end{aligned} \tag{25}$$

Thus,

$$\begin{aligned}&\frac{1}{1+\beta_2}\left(\widetilde{\mathcal{F}}(\mathbf{x}^*)+\kappa\right) \leq \mathcal{F}^l(\mathbf{x}^*)+\kappa \leq \max_{\mathbf{x}}\left(\mathcal{F}^l(\mathbf{x})+\kappa\right) \leq \frac{1+\beta_1}{1-\epsilon}(\widetilde{\mathcal{F}}(\mathbf{x}^*)+\kappa) \\ \implies &\frac{\mathcal{F}^l(\mathbf{x}^*)+\kappa}{\max_{\mathbf{x}}\{\mathcal{F}^l(\mathbf{x})+\kappa\}} \geq \frac{1-\epsilon}{(1+\beta_1)(1+\beta_2)},\end{aligned} \tag{26}$$

which further leads to:

$$\mathcal{F}^l(\mathbf{x}^*) \geq \frac{(1-\epsilon)\max_{\mathbf{x}}\{\mathcal{F}^l(\mathbf{x})\}}{(1+\beta_1)(1+\beta_2)} - \kappa\frac{\epsilon+\beta_1+\beta_2+\beta_1\beta_2}{(1+\beta_1)(1+\beta_2)},$$

as desired.

For the case of additive error $\widetilde{\mathcal{F}}(\mathbf{x}^*) \geq \max_{\mathbf{x}} \widetilde{\mathcal{F}}(\mathbf{x}) - \epsilon$, similarly, we can write:

$$\begin{aligned}\frac{1}{1+\beta_2}\left(\widetilde{\mathcal{F}}(\mathbf{x}^*)+\kappa\right) &\leq \mathcal{F}^l(\mathbf{x}^*)+\kappa \leq \max_{\mathbf{x}}\left(\mathcal{F}^l(\mathbf{x})+\kappa\right) \\ &\leq (1+\beta_1)\max_{\mathbf{x}}\left(\widetilde{\mathcal{F}}(\mathbf{x})+\kappa\right) \leq (1+\beta_1)\left(\widetilde{\mathcal{F}}(\mathbf{x}^*)+\epsilon+\kappa\right),\end{aligned}$$

which yields:

$$\begin{aligned}\frac{\mathcal{F}^l(\mathbf{x}^*)+\kappa}{\max_{\mathbf{x}}\{\mathcal{F}^l(\mathbf{x})+\kappa\}} &\geq \frac{1}{(1+\beta_1)(1+\beta_2)}\frac{1}{1+\epsilon/(\widetilde{\mathcal{F}}(\mathbf{x}^*)+\kappa)} \geq \frac{1}{\eta} \\ \Rightarrow \mathcal{F}^l(\mathbf{x}^*) &\geq \frac{1}{\eta}\max_{\mathbf{x}}\{\mathcal{F}^l(\mathbf{x})\} - \kappa\frac{\eta-1}{\eta},\end{aligned}$$

as desired, which completes our proof.

9.7 Proof of Lemma 1

Proof. Remind that we have β_1 and β_2 defined as follows:

$$\beta_1 = \max_{\mathbf{x}}\max_{s\in\mathcal{L}}\left\{\frac{\sum_{\tau\in\Delta^+(s)}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)}{\sum_{\tau\in\Delta(s)}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)}\right\}, \quad \beta_2 = \max_{\mathbf{x}}\left\{\frac{\sum_{\tau\in\bigcup_s\{\Delta^+(s)\}}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)}{\sum_{\tau\in\bigcup_s\{\Delta(s)\}}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)}\right\},$$

For any defender strategy $x \in \mathcal{X}$, we can re-write the defender's expected utility as follows:

$$\mathcal{F}^l(\mathbf{x}) = \frac{\sum_{s\in\mathcal{L}} r^l(s,x_s)\left(\sum_{\tau\in\Delta(s)}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)+\sum_{\tau\in\Delta^+(s)}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)\right)}{\sum_{\tau\in\bigcup_s\{\Delta(s)\}}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)+\sum_{\tau\in\bigcup_s\{\Delta^+(s)\}}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)} \stackrel{\text{def}}{=} \frac{\mathcal{U}}{\mathcal{V}} \tag{27}$$

According to the definition of β_1, we obtain:

$$\begin{aligned}&\sum_{\tau\in\Delta^+(s)}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right) \leq \beta_1\sum_{\tau\in\Delta(s)}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right) \\ \implies &\mathcal{U} \leq \sum_{s\in\mathcal{L}} r^l(s,x_s)\left(\sum_{\tau\in\Delta(s)}(1+\beta_1)\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)\right)\end{aligned}$$

In addition, we have: $\mathcal{V} \geq \sum_{\tau \in \bigcup_s \{\Delta(s)\}} \exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)$. As a result, we obtain the following inequality:

$$\mathcal{F}^l(\mathbf{x}) \leq \frac{\sum_{s\in\mathcal{L}} r^l(s,x_s)\left(\sum_{\tau\in\Delta(s)}(1+\beta_1)\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)\right)}{\sum_{\tau\in\bigcup_s\{\Delta(s)\}}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)} = (1+\beta_1)\widetilde{\mathcal{F}}(\mathbf{x}) \qquad (*)$$

On the other hand, according to the definition of β_2, we obtain:

$$\sum_{\tau\in\bigcup_s\{\Delta^+(s)\}}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right) \leq \beta_2 \sum_{\tau\in\bigcup_s\{\Delta(s)\}}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)$$
$$\Longrightarrow \mathcal{V} \leq \sum_{\tau\in\bigcup_s\{\Delta(s)\}}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)(1+\beta_2)$$

In addition, we have: $\mathcal{U} \geq \sum_{s\in\mathcal{L}} r^l(s,x_s)\left(\sum_{\tau\in\Delta(s)}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)\right)$. As a result, we obtain:

$$\mathcal{F}^l(\mathbf{x}) \geq \frac{\sum_{s\in\mathcal{L}} r^l(s,x_s)\left(\sum_{\tau\in\Delta(s)}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)\right)}{\sum_{\tau\in\bigcup_s\{\Delta(s)\}}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)(1+\beta_2)} = \frac{1}{1+\beta_2}\widetilde{\mathcal{F}}(\mathbf{x}) \qquad (**)$$

The combination of (*) and (**) concludes our proof. □

9.8 Proof of Lemma 2

Proof. We reuse the definitions of $\mathcal{U}$ and $\mathcal{V}$ as in the proof of Lemma 1. Similar to proof of Lemma 1, according to the definition of β_1, we obtain:

$$\sum_{\tau\in\Delta^+(s)}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right) \leq \beta_1 \sum_{\tau\in\Delta(s)}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)$$

Besides, according to the definition $\kappa = \sum_{s\in\mathcal{L}}\max_{\mathbf{x}}\left|r^l(s,x_s)\right|$, we have $r^l(s,x_s)+\kappa \geq 0$ for any $\mathbf{x}\in\mathcal{X}$. Thus, we can write:

$$\mathcal{U}+\kappa\mathcal{V} = \sum_{s\in\mathcal{L}}\left(r^l(s,x_s)+\kappa\right)\left(\sum_{\tau\in\Delta(s)\bigcup\Delta^+(s)}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)\right)$$
$$\leq \sum_{s\in\mathcal{L}}\left(r^l(s,x_s)+\kappa\right)\left(\sum_{\tau\in\Delta(s)}(1+\beta_1)\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)\right)$$

In addition, we have: $\mathcal{V} \geq \sum_{\tau\in\bigcup_s\{\Delta(s)\}}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)$. As a result, we obtain the following inequality:

$$\mathcal{F}^l(\mathbf{x})+\kappa = \frac{\mathcal{U}+\kappa\mathcal{V}}{\mathcal{V}} \leq \frac{\sum_{s\in\mathcal{L}}\left(r^l(s,x_s)+\kappa\right)\left(\sum_{\tau\in\Delta(s)}(1+\beta_1)\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)\right)}{\sum_{\tau\in\bigcup_s\{\Delta(s)\}}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)}$$
$$= (1+\beta_1)\left(\widetilde{\mathcal{F}}(\mathbf{x})+\kappa\right) \qquad (28)$$

On the other hand, from the way we select κ, we have:

$$\kappa \geq \sum_{s\in\mathcal{L}}|r^l(s,x_s)| \geq \sum_{s\in\mathcal{L}}|r^l(s,x_s)|\frac{\sum_{\tau\in\Delta(s)}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)}{\sum_{\tau\in\bigcup_{s'}\{\Delta(s')\}}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)}$$
$$\Longrightarrow \sum_{s\in\mathcal{L}}\left(r^l(s,x_s)+\kappa\right)\left(\sum_{\tau\in\Delta(s)}\exp\left(\frac{U(\tau;\mathbf{x}))}{\mu}\right)\right) \geq 0$$

Besides, according to the definition of β_2, we obtain:

$$\sum_{\tau \in \bigcup_s \{\Delta^+(s)\}} \exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right) \leq \beta_2 \sum_{\tau \in \bigcup_s \{\Delta(s)\}} \exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)$$

$$\Longrightarrow \mathcal{V} \leq \sum_{\tau \in \bigcup_s \{\Delta(s)\}} \exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)(1+\beta_2)$$

As a result, we obtain the following inequalities,

$$\begin{aligned}\mathcal{F}^l(\mathbf{x}) + \kappa &\geq \frac{\sum_{s\in\mathcal{L}}(r^l(s,x_s)+\kappa)\left(\sum_{\tau\in\Delta(s)}\exp\left(\frac{U(\tau;\mathbf{x}))}{\mu}\right)\right)}{\mathcal{V}} \\ &\geq \frac{\sum_{s\in\mathcal{L}}(r^l(s,x_s)+\kappa)\left(\sum_{\tau\in\Delta(s)}\exp\left(\frac{U(\tau;\mathbf{x}))}{\mu}\right)\right)}{\sum_{\tau\in\bigcup_s\{\Delta(s)\}}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)(1+\beta_2)} = \frac{1}{1+\beta_2}(\widetilde{\mathcal{F}}(\mathbf{x})+\kappa). \end{aligned} \tag{29}$$

Combining (28) and (29) gives us the desired inequalities. □

10 Solving the Restricted Problem in Section 6.3

In order to solve the restricted problem, we also propose to apply the binary search approach. The resulting sub-problem at each binary search step can be formulated as follows:

$$\begin{aligned}\max_{\mathbf{x}\in\mathcal{X}} \widetilde{g}(\mathbf{x},\lambda) = &\sum_{s\in\mathcal{L}}\sum_{\tau\in\Delta(s)} r^l(s,x_s)\exp\Big(\frac{U(\tau;\mathbf{x})}{\mu}\Big) \\ &- \lambda\Big[\sum_{s\in\mathcal{L}}\sum_{\tau\in\Delta(s)}\exp\Big(\frac{U(\tau;\mathbf{x})}{\mu}\Big)\Big] \end{aligned} \tag{sub-Approx}$$

In the following, we present our theoretical results on the key underlying property of (sub-Approx), as well as new exact solutions for solving (sub-Approx).

10.1 Unimodality

Our Theorem 3 shows that we can use a gradient-based method to obtain the unique global optimal solution to (sub-Approx).

Theorem 3. (sub-Approx) *is unimodal; any local optimal solution* $\boldsymbol{x}^*$ *of* (sub-Approx) *is also globally optimal.*

Proof. The proof can be divided into two major steps:

Step 1: showing that (sub-Approx) **can be converted into a (strictly) convex optimization problem.** This step is done via variable transformation. Recall that the adversary utility $v(s;\mathbf{x}) = v(s,x_s) = w^f x_s + t^f$ and defender utility $r^l(s;\mathbf{x}) = r^l(s,x_s) = w^l x_s + t^l$ where $w^f < 0$ and $w_l > 0$. Essentially, we introduce a new variable $y_s = \exp\left(\frac{v(s,x_s)}{\mu}\right)$ for all critical nodes $s \in \mathcal{L}$. We will show that the objective $\widetilde{g}(\mathbf{x},\lambda)$ of (sub-Approx) can be rewritten as a strictly concave function of $\{y_s\}$.

For each trajectory $\tau \in \Delta(s)$, let $V^s(\tau) = \sum_{s'\in\tau,\ s'\neq s} v(s';\mathbf{x})$, or equivalently, $V^s(\tau) = U(\tau;\mathbf{x}) - v(s;\mathbf{x})$, which is the accumulated adversary utility over every node on τ except node s. We see that $V^s(\tau)$ is independent of the defender coverage probability x_s at every critical node $s \in \mathcal{L}$ (by definition of $\Delta(s)$). Therefore, we can reformulate the objective $\widetilde{g}(\mathbf{x},\lambda)$ of (sub-Approx) as follows:

$$\widetilde{g}(\mathbf{x},\lambda) = \sum_{s\in\mathcal{L}} r^l(s,x_s)\exp\left(\frac{v(s;\mathbf{x})}{\mu}\right)H(s) - \lambda\left(\sum_{s\in\mathcal{L}}\exp\left(\frac{v(s;\mathbf{x})}{\mu}\right)H(s)\right).$$

where $H(s) = \sum_{\tau\in\Delta(s)}\exp(V^s(\tau)/\mu)$. We thus can write $\widetilde{g}(\mathbf{x},\lambda)$ as function of $\mathbf{y}$ as follows:

$$\begin{aligned}\widetilde{g}(\mathbf{y},\lambda) &= \sum_{s\in\mathcal{L}}\left(\left(\mu\log(y_s) - t^f\right)\frac{w^l}{w^f} + t^l\right)y_s H(s) - \lambda\left(\sum_{s\in\mathcal{L}} y_s H(s)\right) \\ &= \sum_{s\in\mathcal{L}}\mu\frac{w^l}{w^f}H(s)\log(y_s)y_s - y_s H(s)\left(\mu\frac{t^f w^l}{w^f} + t^l + \lambda\right). \end{aligned} \tag{30}$$

Since $w^f/w^l \leq 0$, it can be shown that each component $\mu\frac{w^l}{w^f}H(s)\log(y_s)y_s$ is concave in y_s, thus $\widetilde{g}(\mathbf{y},\lambda)$ is strictly concave in $\mathbf{y}$ for all critical nodes s. Moreover, for any $k \in [K]$, the constraint $\sum_{s\in\mathcal{L}_k} x_s \leq M_k$ becomes $\sum_{s\in\mathcal{L}_k}\frac{\mu\log(y_s)-t^f}{w^f} \leq M_k$, which is convex since $w^f < 0$.

Step 2: proving global optimality via the KKT condition correspondence with variable transformation Under the variable transformation as presented in **Step 1**, for notational convenience, let us define $\widehat{x}_s(\cdot) : \mathbb{R} \to \mathbb{R}$ and $\widehat{y}_s(\cdot) : \mathbb{R} \to \mathbb{R}$ such that:

$$\widehat{y}_s(x_s) = \exp\left(\frac{v(s, x_s)}{\mu}\right)$$
$$\widehat{x}_s(y_s) = \frac{\mu \log(y_s) - t^f}{w^f},$$

i.e., the mappings from $\mathbf{x}_s$ to $\mathbf{y}_s$ and vice-versa.

Recall that the feasible strategy space of the defender $\mathcal{X} = \left\{\mathbf{x} : \sum_{s\in\mathcal{L}_k} x_s \le M_k, \forall k, x_s \in [L^x, U^x]\right\}$. We thus can write the Lagrange dual of (sub-Approx) as follows:

$$L^g(\mathbf{x}, \boldsymbol{\gamma}, \boldsymbol{\eta}^1, \boldsymbol{\eta}^2) = \widetilde{g}(\mathbf{x}, \lambda) - \sum_k \gamma_k \left(\sum_{s\in\mathcal{L}_k} x_s - M_k\right) - \sum_s \eta^1_s (x_s - U^x) + \sum_s \eta^2 (x_s - L^x).$$

Since $\mathbf{x}^*$ is a local optimal solution for (sub-Approx), the KKT conditions imply that there are dual $\boldsymbol{\gamma}^*, \boldsymbol{\eta}^{1*}, \boldsymbol{\eta}^{1*} \ge 0$ such that the following constraints are satisfied:

$$\begin{cases} \frac{\widetilde{g}(\mathbf{x}^*,\lambda)}{\partial x_s} - \gamma^*_k - \eta^{1*}_s + \eta^{2*}_s = 0, \text{ where } k \text{ such that } s \in \mathcal{L}_k \\ \gamma^*_k \left(\sum_{s\in\mathcal{L}_k} x_s - M_k\right) = 0, \ \forall k \\ \eta^{1*}_s (x^*_s - U^x) = 0 \\ \eta^{2*}_s (x^*_s - L^x) = 0 \\ L^x \le x^*_s \le U^x \\ \sum_{s\in\mathcal{L}_k} x_s - M_k, \ \forall k \end{cases} \tag{31}$$

By the variance transformation $y_s = \exp\left(\frac{(w^f x_s + t^f)}{\mu}\right)$, let $y^*_s = \exp\left(\frac{(w^f x^*_s + t^f)}{\mu}\right)$ and $x_s = \widehat{x}_s(y_s) = \frac{\mu \log(y_s) - t^f}{w^f}$ for all $s \in \mathcal{L}$, we can write (31) equivalently as:

$$\begin{cases} \frac{\widetilde{g}(\mathbf{x}^*,\lambda)}{\partial x_s} \frac{\partial \widehat{x}_s(y^*_s)}{\partial y_s} - \gamma^*_k \frac{\partial \widehat{x}_s(y^*_s)}{\partial y_s} - \eta^{1*}_s \frac{\partial \widehat{x}_s(y^*_s)}{\partial y_s} + \eta^{2*}_s \frac{\partial \widehat{x}_s(y^*_s)}{\partial y_s} = 0 \\ \gamma^*_k \left(\sum_{s\in\mathcal{L}} \frac{\mu \log(y^*_s) - t^f}{w^f} - M\right) = 0, \ \forall k \\ \eta^{1*}_s \left(\frac{\mu \log(y^*_s) - t^f}{w^f} - U^x)\right) = 0 \\ \eta^{2*}_s \left(\frac{\mu \log(y^*_s) - t^f}{w^f} - L^x\right) = 0 \\ L^x \le \frac{\mu \log(y^*_s) - t^f}{w^f} \le U^x \\ \sum_{s\in\mathcal{L}_k} \frac{\mu \log(y^*_s) - t^f}{w^f} \le M_k, \ \forall k. \end{cases} \tag{32}$$

The first condition of (32) can be written equivalently as follows:

$$\frac{\widetilde{g}(\mathbf{y}^*, \lambda)}{\partial y_s} - \gamma^*_k \mathbb{I}[s \in \mathcal{L}_k] \frac{\partial \widehat{x}_s(y^*_s)}{\partial y_s} - \eta^{1*}_s \frac{\partial \widehat{x}_s(y^*_s)}{\partial y_s} + \eta^{2*}_s \frac{\partial \widehat{x}_s(y^*_s)}{\partial y_s} = 0,$$

where $\mathbb{I}[\cdot]$ is the indicator function. This implies that $\mathbf{y}^*, \boldsymbol{\gamma}^*, \boldsymbol{\eta}^{1*}, \boldsymbol{\eta}^{1*}$ also satisfy the KKT conditions of the following (strictly) convex optimization problem (i.e., the resulting problem of variable transformation discussed in **Step 1**).

$$\begin{aligned} \max_{\mathbf{y}} \quad & \widetilde{g}(\mathbf{y}, \lambda) \\ \text{subject to} \quad & \sum_{s\in\mathcal{L}_k} \widehat{x}_s(y_s) \le M_k, \ \forall k \text{ and } \widehat{x}_s(y_s) \in [L^x, U^x]. \end{aligned} \tag{33}$$

Thus, $\mathbf{y}^*$ is the unique global optimal solution to (33), which also means that $\mathbf{x}^*$ is also the global optimal solution to (sub-Approx) as desired. □

10.2 Exact Solution.

Even though we can use a gradient-based method to obtain the unique global optimal solution to (sub-Approx), thanks to the unimodality property shown above, the computational challenge is that the objective $\widetilde{g}(\mathbf{x}, \lambda)$ still involves exponentially many paths. Our idea is to decompose $\widetilde{g}(\mathbf{x}, \lambda)$ into multiple terms (each term corresponds to a critical node $s \in \mathcal{L}$) — which can be computed using dynamic programming. Essentially, we create new graphs by keeping a node $s \in \mathcal{L}$ and remove every other nodes in $\mathcal{L}$. Let $\mathcal{G}(s)$ be the sub-graph created from the graph $\mathcal{G}$ by deleting all nodes in $\mathcal{L}$ except node s. Since we will be dealing with several graphs, henceforth we denote all paths in any arbitrary graph $\mathcal{G}$ as $\Omega(\mathcal{G})$. The proposition below shows that $\widetilde{g}(\mathbf{x}, \lambda)$ can be decomposed into terms that can be efficiently computed based on sub-graphs $\mathcal{G}(s), \forall s \in \mathcal{S}$.

Proposition 5. *$\widetilde{g}(\boldsymbol{x}, \lambda)$ can be written as follows:*

$$\sum_{s\in\mathcal{L}}\Big(\sum_{\substack{\tau\ni s\\ \tau\in\Omega(\mathcal{G}(s))}}\Big[r^l(s,x_s)\exp\Big(\frac{U(\tau;\boldsymbol{x})}{\mu}\Big)-\lambda\exp\Big(\frac{U(\tau;\boldsymbol{x})}{\mu}\Big)\Big]\Big) \tag{34}$$

The proof is straightforward, so we'll skip the details. Given any sub-graph $\mathcal{G}(s)$, the terms in $\widetilde{g}(\mathbf{x}, \lambda)$ (and their gradients) can be computed by solving a system of linear equations, similarly as the approach described in in Algorithm 1.

10.3 Efficient Approximation Solution for the Restricted Problem

The exact method mentioned in Section 10.2 above might not scale up when $|\mathcal{L}|$ is large. In this section, we thus propose a new approach which is both efficient and guarantees a bounded approximate solution for (OPT). Our main ideas can be summarized as follows: (a) we identify a graph modification and solve (OPT) with the modified graph using the Algorithm 1; and (b) we theoretically shows that this resulting solution (obtained from (a)) is also a bounded approximate solution for (OPT). We elaborate our ideas in the following.

Network modification. We modify the network $\mathcal{G}$ by raising the costs of travelling between any pair of nodes in $\mathcal{L}$ in such a way that β_1 and β_2 become arbitrarily small. We remark that, given any $\epsilon' > 0$, we can always modify the travelling costs between pairs of nodes in $\mathcal{L}$ to obtain a modified network $\mathcal{G}'$ such that the following conditions holds:

$$\max_{\mathbf{x}}\max_{s\in\mathcal{L}}\Big\{\frac{\sum_{\tau\in\Delta^+(s)}\exp\left({}^{U(\tau;\mathbf{x})}/_{\mu}\right)}{\sum_{\tau\in\Omega,\tau\ni s}\exp\left({}^{U(\tau;\mathbf{x})}/_{\mu}\right)}\Big\}\leq\epsilon' \tag{35}$$

$$\max_{\mathbf{x}}\Big\{\frac{\sum_{\tau\in\bigcup_s\{\Delta^+(s)\}}\exp\left({}^{U(\tau;\mathbf{x})}/_{\mu}\right)}{\sum_{\tau\in\Omega}\exp\left({}^{U(\tau;\mathbf{x})}/_{\mu}\right)}\Big\}\leq\epsilon'. \tag{36}$$

We remind that $\Delta^+(s)$ be the set of paths that cross s and at least another node in $\mathcal{L}$ and Ω is the set of all paths. We denote the objective of (13) of the binary search step for (OPT) with respect to the modified network $\mathcal{G}'$ as $g(\mathbf{x}, \lambda|\mathcal{G}')$. We can optimize $g(\mathbf{x}, \lambda|\mathcal{G}')$ by running gradient descent. The problem $\max_{\mathbf{x}} g(\mathbf{x}, \lambda|\mathcal{G}')$ is not convex, yet we can provide the following strong guarantee.

Solution theoretical bounds. Let us first define:

$$\rho^s=\sum\nolimits_{\tau\in\Omega,\tau\ni s}\exp\Big(\frac{U(\tau;L^x\mathbf{e})}{\mu}\Big),\ \forall s\in\mathcal{L}$$
$$\rho=\sum\nolimits_{\tau\in\Omega}\exp\Big(\frac{U(\tau;L^x\mathbf{e})}{\mu}\Big)$$

where $\mathbf{e}$ is an all-one vector of size $|\mathcal{L}|$. We remind that L^x and U^x are the lower and upper bounds on the resource coverage x_s at every critical node $s \in \mathcal{L}$. We present our theoretical bound w.r.t our original problem (OPT) (the proof is presented in the next section for the sake of clarity).

Theorem 4. *If we run binary search to solve* (OPT) *with respect to the modified network $\mathcal{G}'$ and obtain $(\overline{\boldsymbol{x}}, \overline{\lambda})$, the following performance bound is guaranteed:*

$$\mathcal{F}^l(\overline{\boldsymbol{x}})\geq\left({}^{1}/_{\overline{\eta}}\right)\max_{\boldsymbol{x}}\{\mathcal{F}^l(\boldsymbol{x})\}-\kappa\left({}^{\overline{\eta}-1}/_{\overline{\eta}}\right),$$

where $\overline{\eta}=(1+\beta_1)(1+\beta_2)\Big(1+\epsilon'\mathcal{U}\Big)$, where $\mathcal{U}$ is a constant independent of $\epsilon', \beta_1, \beta_2$.

Finally, Algorithm 2 describes the approximation scheme.

Algorithm 2: *Solving* (OPT) *through the restricted interdiction problem* (Approx-OPT)

- Create the new graph $\mathcal{G}'$ by raising the traveling cost between any pair of nodes in $\mathcal{L}$.
- Solve (OPT) using Alg. 1 with $\mathcal{G}'$.
- Recover the original graph $\mathcal{G}$ and re-solve (OPT) to improve the solution obtained from the above step.

Remark 2. *If β_1, β_2 and ϵ' are small, then $\overline{\eta}$ would be close to 1 and $\overline{\boldsymbol{x}}$ would be close to an optimal solution to* (OPT)*. Note that β_1, β_2 can be small in a real situation where the costs of traveling between critical nodes are expensive to the adversary (e.g., critical nodes are far away from each other).*

Proof of Theorem 4

To prove the result, we need the folllowing lemmas:

Lemma 3. *For any $(\boldsymbol{x}, \lambda)$, then we have:*

$$|g(\boldsymbol{x}, \lambda|\mathcal{G}') - \widetilde{g}(\boldsymbol{x}, \lambda)| \leq \epsilon' \left(\kappa \max_s\{\rho^s\} + \lambda\rho\right).$$

Proof. Observing that $\{\tau; \tau \in \Omega, \tau \ni s\} = \Delta^+(s) \cup \Delta(s)$ and $\Delta^+(s), \Delta(s)$ are disjoint, we can decompose $g(\mathbf{x}, \lambda|\mathcal{G}')$ into two separate terms, as follows:

$$\begin{aligned} g(\mathbf{x}, \lambda|\mathcal{G}') &= \sum_{s\in\mathcal{L}} \sum_{\substack{\tau\in\Omega \\ \tau\ni s}} r_s^f(x_s) \exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right) - \lambda \left(\sum_{\tau'\in\Omega} \exp\left(\frac{U(\tau';\mathbf{x})}{\mu}\right)\right) \\ &= \widetilde{g}(\mathbf{x}, \lambda) + \mathcal{T}(\mathbf{x}, \lambda), \end{aligned} \tag{37}$$

where the second term:

$$\mathcal{T}(\mathbf{x}, \lambda) = \sum_{s\in\mathcal{L}} r_s^f(x_s) \sum_{\tau\in\Delta^+(s)} \exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right) - \lambda \left(\sum_{\tau\in\bigcup_{s\in\mathcal{L}}\Delta^+(s)} \exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)\right).$$

Moreover, remind that we have the definition of ρ^s and ρ:

$$\rho^s = \sum_{\tau\in\Omega,\tau\ni s} \exp\left(\frac{U(\tau; L^x\mathbf{e})}{\mu}\right),\ \forall s \in \mathcal{L} \qquad\qquad \rho = \sum_{\tau\in\Omega} \exp\left(\frac{U(\tau; L^x\mathbf{e})}{\mu}\right)$$

According to conditions in Equation (35) and (36), we obtain:

$$\max_{\mathbf{x}} \max_{s\in\mathcal{L}} \left\{ \frac{\sum_{\tau\in\Delta^+(s)} \exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)}{\sum_{\tau\in\Omega,\tau\ni s} \exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)} \right\} \leq \epsilon' \implies \sum_{\tau\in\Delta^+(s)} \exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right) \leq \epsilon'\rho^s,\ \forall \mathbf{x} \in \mathcal{X}, s \in \mathcal{L} \tag{38}$$

$$\max_{\mathbf{x}} \left\{ \frac{\sum_{\tau\in\bigcup_s\{\Delta^+(s)\}} \exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)}{\sum_{\tau\in\Omega} \exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)} \right\} \leq \epsilon' \implies \sum_{\tau\in\bigcup_s\{\Delta^+(s)\}} \exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right) \leq \epsilon'\rho,\ \forall \mathbf{x} \in \mathcal{X}. \tag{39}$$

By using these inequalities, we have:

$$|\mathcal{T}(\mathbf{x}, \lambda)| \leq \sum_{s\in\mathcal{L}} |r_s^f(x_s)|\epsilon'\rho^s + \lambda\epsilon'\rho \leq \epsilon' \left(\kappa \max_s\{\rho^s\} + \lambda\rho\right). \tag{40}$$

which concludes our proof. □

Given λ, Lemma 4 below shows that any local optimal solution to $\max_{\mathbf{x}} g(\mathbf{x}, \lambda|\mathcal{G}')$ (i.e., the binary step of (OPT) with modified $\mathcal{G}'$) is in an $\mathcal{O}(\epsilon')$ neighborhood of optimal solutions to (sub-Approx): $\max_{\mathbf{x}} \widetilde{g}(\mathbf{x}, \lambda)$, i.e., the binary step of the restricted problem (Approx-OPT) with original graph $\mathcal{G}$.

Lemma 4. *Let $\bar{\boldsymbol{x}}$ be a local optimal solution of $\max_{\boldsymbol{x}} g(\boldsymbol{x}, \lambda|\mathcal{G}')$ for a given λ, then we have:*

$$\max_{\boldsymbol{x}}\{\widetilde{g}(\boldsymbol{x}, \lambda)\} - \widetilde{g}(\bar{\boldsymbol{x}}, \lambda) \leq \epsilon'\mathcal{H},$$

where $\mathcal{H} = 2\left(\mu\rho^j + \frac{\kappa}{|w_j^f|}\max_s\{\rho^s\} + \lambda\rho\right)\exp\left(\frac{w_j^f(L^x - U^x)}{\mu}\right)$.

Proof. We reuse the decomposition of $g(\mathbf{x}, \lambda|\mathcal{G}')$ as described in Lemma 3. By taking the derivative of $\mathcal{T}(\mathbf{x}, \lambda)$ w.r.t x_j, $j \in \mathcal{L}$, we obtain:

$$\begin{aligned} \frac{\partial\mathcal{T}(\mathbf{x}, \lambda)}{\partial x_j} &= w_j^f \sum_{\tau\in\Delta^+(j)} \exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right) + \frac{1}{\mu}\sum_{s\in\mathcal{L}} r_s^f(x_s) \sum_{\tau\in\Delta^+(s),\tau\ni j} w_j^f \exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right) \\ &\quad - \frac{\lambda}{\mu}\left(\sum_{\tau\in\bigcup_{s\in\mathcal{L}}\Delta^+(s),\tau\ni j} w_j^f \exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)\right). \end{aligned}$$

Thus, using (38) and (39), we get the inequality:

$$\left|\frac{\partial \mathcal{T}(\mathbf{x},\lambda)}{\partial x_j}\right| \leq \epsilon' \left(|w_j^f|\rho^j + \frac{\kappa}{\mu}\max_s\{\rho^s\} + \frac{\lambda}{\mu}|w_j^f|\rho\right). \tag{41}$$

Now, let us consider the problem $\max_{\mathbf{x}}\{g(\mathbf{x},\lambda|\mathcal{G}')|\ \mathbf{x}\in\mathcal{X}\}$. We form its Lagrange dual as:

$$L^{\mathcal{G}'}(\mathbf{x},\boldsymbol{\gamma},\boldsymbol{\eta}) = g(\mathbf{x},\lambda|\mathcal{G}') - \sum_{k\in[K]}\gamma_k\left(\sum_{s\in\mathcal{L}_k}x_s - M_k\right) - \sum_s \eta_s^1(x_s - U^x) + \sum_s \eta_s^2(x_s - L^x).$$

If $\overline{\mathbf{x}}$ is a stationary point of $\max_{\mathbf{x}}\{g(\mathbf{x},\lambda|\mathcal{G}')|\ \mathbf{x}\in\mathcal{X}\}$, then the KKT conditions imply that there are $\boldsymbol{\gamma}^*, \boldsymbol{\eta}^{1*}, \boldsymbol{\eta}^{2*}$ such that:

$$\frac{\partial g(\overline{\mathbf{x}},\lambda|\mathcal{G}')}{\partial x_j} - \sum_{k\in[K]}\gamma_k^*\mathbb{I}[j\in\mathcal{L}_k] - \eta_j^{1*} + \eta_j^{2*} = 0,$$

where $\mathbb{I}[\cdot]$ is the indicator function. Note that $\frac{\partial g(\overline{\mathbf{x}},\lambda|\mathcal{G}')}{\partial x_j} = \frac{\partial \widetilde{g}(\overline{\mathbf{x}},\lambda)}{\partial x_j} + \frac{\partial \mathcal{T}(\mathbf{x},\lambda)}{\partial x_j}$. Thus, from (41) we have:

$$\begin{aligned}
&\left|\frac{\partial \widetilde{g}(\overline{\mathbf{x}},\lambda)}{\partial x_j} - \sum_{k\in[K]}\gamma_k^*\mathbb{I}[j\in\mathcal{L}_k] - \eta_j^{1*} + \eta_j^{2*}\right| \\
&\leq \left|\frac{\partial g(\overline{\mathbf{x}},\lambda|\mathcal{G}')}{\partial x_j} - \sum_{k\in[K]}\gamma_k^*\mathbb{I}[j\in\mathcal{L}_k] - \eta_j^{1*} + \eta_j^{2*}\right| + \left|\frac{\partial \mathcal{T}(\mathbf{x},\lambda)}{\partial x_j}\right| \\
&\leq \epsilon'\left(|w_j^f|\rho^j + \frac{\kappa}{\mu}\max_s\{\rho^s\} + \frac{\lambda}{\mu}|w_j^f|\rho\right).
\end{aligned} \tag{42}$$

Let us now define a function $\widehat{G}(\mathbf{x},\lambda)$ as follows:

$$\widehat{G}(\mathbf{x},\lambda) = \widetilde{g}(\mathbf{x},\lambda) + \sum_{s\in\mathcal{L}}\alpha_s\exp\left(\frac{w_s^f x_s + t_s^f}{\mu}\right),$$

where $\alpha_s, \forall s\in\mathcal{L}$, are chosen as follows:

$$\alpha_s = -\frac{\frac{\partial \widetilde{g}(\overline{\mathbf{x}},\lambda)}{\partial x_s} - \sum_{k\in[K]}\gamma_k^*\mathbb{I}[j\in\mathcal{L}_k] - \eta_j^{1*} + \eta_j^{2*}}{\frac{w_s^f}{\mu}\exp\left(\frac{w_s^f x_s + t_s^f}{\mu}\right)}.$$

Given α_s defined as above, we obtain the following equations:

$$\frac{\partial \widehat{G}(\overline{\mathbf{x}},\lambda)}{\partial x_j} - \sum_{k\in[K]}\gamma_k^*\mathbb{I}[j\in\mathcal{L}_k] - \eta_j^{1*} + \eta_j^{2*} = 0,\ \forall j\in\mathcal{L}. \tag{43}$$

In the following, we first attempt to bound the gap $\left|\widehat{G}(\mathbf{x},\lambda) - \widetilde{g}(\mathbf{x},\lambda)\right|$ for every defender strategy $\mathbf{x}$. We then leverage this bound together with the unimodality of $\widehat{G}(\mathbf{x},\lambda)$ to bound the gap $|\max_{\mathbf{x}\in\mathcal{X}}\widetilde{g}(\mathbf{x},\lambda) - \widetilde{g}(\overline{\mathbf{x}},\lambda)|$. As we show later, the unimodality of $\widehat{G}(\mathbf{x},\lambda)$ is proved based on Equation 43 and the variable conversion trick used in the proof of Theorem 3.

Bounding the gap $\left|\widehat{G}(\mathbf{x},\lambda) - \widetilde{g}(\mathbf{x},\lambda)\right|$. By taking the derivative of $\widehat{G}(\mathbf{x},\lambda)$ w.r.t. x_j, we get:

$$\frac{\partial \widehat{G}(\mathbf{x},\lambda)}{\partial x_j} = \frac{\widetilde{g}(\mathbf{x},\lambda)}{\partial x_j} + \frac{\alpha_j w_j^f}{\mu}\exp\left(\frac{w_j^f + t_j^f}{\mu}\right).$$

Combining with (42)-(43) we can bound $\alpha_j, \forall j\in\mathcal{L}$, as follows:

$$\begin{aligned}
&\left|\frac{\alpha_j w_j^f}{\mu}\exp\left(\frac{w_j^f\overline{x}_j + t_j^f}{\mu}\right)\right| = \left|\frac{\partial \widehat{G}(\overline{\mathbf{x}},\lambda)}{\partial x_j} - \frac{\widetilde{g}(\overline{\mathbf{x}},\lambda)}{\partial x_j}\right| \\
&\leq \left|\frac{\partial \widehat{G}(\overline{\mathbf{x}},\lambda)}{\partial x_j} - \sum_{k\in[K]}\gamma_k^*\mathbb{I}[j\in\mathcal{L}_k] - \eta_j^{1*} + \eta_j^{2*}\right| + \left|\frac{\partial \widetilde{g}(\overline{\mathbf{x}},\lambda)}{\partial x_j} - \sum_{k\in[K]}\gamma_k^*\mathbb{I}[j\in\mathcal{L}_k] - \eta_j^{1*} + \eta_j^{2*}\right| \\
&\leq \epsilon'\left(|w_j^f|\rho^j + \frac{\kappa}{\mu}\max_s\{\rho^s\} + \frac{\lambda}{\mu}|w_j^f|\rho\right)
\end{aligned} \tag{44}$$

which implies:

$$\left|\alpha_j \exp\left(\frac{w_j^f \overline{x}_j + t_j^f}{\mu}\right)\right| \leq \epsilon' \mu \left(\rho^j + \frac{\kappa}{\mu |w_j^f|} \max_s\{\rho^s\} + \frac{\lambda}{\mu}\rho\right). \tag{45}$$

Combining the above inequality with the definition of $\widehat{G}(\mathbf{x}, \lambda)$ we obtain the following inequality for all defender strategy $\mathbf{x} \in \mathcal{X}$:

$$\begin{aligned}
\left|\widehat{G}(\mathbf{x}, \lambda) - \widetilde{g}(\mathbf{x}, \lambda)\right| &\leq \sum_{j \in \mathcal{L}} \left|\alpha_j \exp\left(\frac{(w_j^f x_j + t_j^f)}{\mu}\right)\right| \\
&\leq \sum_{j \in \mathcal{L}} \left|\alpha_j \exp\left(\frac{w_j^f \overline{x}_j + t_j^f}{\mu}\right) \exp\left(\frac{w_j^f (x_j - \overline{x}_j)}{\mu}\right)\right| \\
&\leq \epsilon' \left(\mu\rho^j + \frac{\kappa}{|w_j^f|} \max_s\{\rho^s\} + \lambda\rho\right) \exp\left(\frac{w_j^f (L^x - U^x)}{\mu}\right).
\end{aligned}$$

Bounding $|\max_{\mathbf{x} \in \mathcal{X}} \widetilde{g}(\mathbf{x}, \lambda) - \widetilde{g}(\overline{\mathbf{x}}, \lambda)|$. Let us define $\mathcal{H} = 2\left(\mu\rho^j + \frac{\kappa}{|w_j^f|} \max_s\{\rho^s\} + \lambda\rho\right) \exp\left(\frac{w_j^f (L^x - U^x)}{\mu}\right)$ for notational simplicity. We have the following remarks:

(i) From (43), we see that $\overline{\mathbf{x}}$ is a stationary point of the maximization problem $\max_{\mathbf{x} \in \mathcal{X}}\{\widehat{G}(\mathbf{x}, \lambda)\}$ and $\boldsymbol{\gamma}^*, \boldsymbol{\eta}^{1*}, \boldsymbol{\eta}^{2*}$ are the corresponding KKT multipliers.

(ii) With the change of variables used in the proof of Theorem 3, the function $\widehat{G}(\mathbf{x}, \lambda)$ becomes $\widetilde{g}(\widehat{\mathbf{x}}(\mathbf{y}), \lambda) + \sum_{s \in \mathcal{L}} \alpha_s y_s$, thus $\widehat{G}(\widehat{\mathbf{x}}(\mathbf{y}), \lambda)$ is strictly concave in $\mathbf{y}$. Similar to Theorem 3, $\widehat{G}(\mathbf{x}, \lambda)$ is unimodal (i.e., any local optimum is a global one).

Thus, $\overline{\mathbf{x}}$ is also an optimal solution to $\max_{\mathbf{x} \in \mathcal{X}} \widehat{G}(\mathbf{x}, \lambda)$. As a result, we now can bound the gap $|\max_{\mathbf{x} \in \mathcal{X}} \widetilde{g}(\mathbf{x}, \lambda) - \widetilde{g}(\overline{\mathbf{x}}, \lambda)|$ as follows:

$$\begin{aligned}
|\max_{\mathbf{x} \in \mathcal{X}} \widetilde{g}(\mathbf{x}, \lambda) - \widetilde{g}(\overline{\mathbf{x}}, \lambda)| &\leq |\max_{\mathbf{x} \in \mathcal{X}} \widetilde{g}(\mathbf{x}, \lambda) - \widehat{G}(\overline{\mathbf{x}}, \lambda)| + |\widehat{G}(\overline{\mathbf{x}}, \lambda) - \widetilde{g}(\overline{\mathbf{x}}, \lambda)| \\
&\leq |\max_{\mathbf{x} \in \mathcal{X}} \widetilde{g}(\mathbf{x}, \lambda) - \max_{\mathbf{x} \in \mathcal{X}} \widehat{G}(\mathbf{x}, \lambda)| + \frac{\epsilon' \mathcal{H}}{2}.
\end{aligned} \tag{46}$$

We consider the following two cases:

- If $\max_{\mathbf{x} \in \mathcal{X}} \widetilde{g}(\mathbf{x}, \lambda) \geq \max_{\mathbf{x} \in \mathcal{X}} \widehat{G}(\mathbf{x}, \lambda)$. Let $\mathbf{x}^*$ be optimal for $\max_{\mathbf{x} \in \mathcal{X}} \widetilde{g}(\mathbf{x}, \lambda)$, we have

$$\begin{aligned}
|\max_{\mathbf{x} \in \mathcal{X}} \widetilde{g}(\mathbf{x}, \lambda) - \max_{\mathbf{x} \in \mathcal{X}} \widehat{G}(\mathbf{x}, \lambda)| &= \widetilde{g}(\mathbf{x}^*, \lambda) - \max_{\mathbf{x} \in \mathcal{X}} \widehat{G}(\mathbf{x}, \lambda) \\
&\leq \widetilde{g}(\mathbf{x}^*, \lambda) - \widehat{G}(\mathbf{x}^*, \lambda) \\
&\leq \frac{\epsilon' \mathcal{H}}{2}
\end{aligned} \tag{47}$$

- If $\max_{\mathbf{x} \in \mathcal{X}} \widetilde{g}(\mathbf{x}, \lambda) < \max_{\mathbf{x} \in \mathcal{X}} \widehat{G}(\mathbf{x}, \lambda)$, then we have

$$\begin{aligned}
|\max_{\mathbf{x} \in \mathcal{X}} \widetilde{g}(\mathbf{x}, \lambda) - \max_{\mathbf{x} \in \mathcal{X}} \widehat{G}(\mathbf{x}, \lambda)| &= \max_{\mathbf{x} \in \mathcal{X}} \widehat{G}(\mathbf{x}, \lambda) - \max_{\mathbf{x} \in \mathcal{X}} \widetilde{g}(\mathbf{x}, \lambda) \\
&\leq \widehat{G}(\overline{\mathbf{x}}, \lambda) - \widetilde{g}(\overline{\mathbf{x}}, \lambda) \\
&\leq \frac{\epsilon' \mathcal{H}}{2}
\end{aligned} \tag{48}$$

Combine (46)-(47)-(48) we obtain:

$$|\max_{\mathbf{x} \in \mathcal{X}} \widetilde{g}(\mathbf{x}, \lambda) - \widetilde{g}(\overline{\mathbf{x}}, \lambda)| \leq \epsilon' \mathcal{H}$$

which is the desired inequality, completing our proof.

□

Lemma 5. *If we run binary search to solve* (OPT) *with the modified network $\mathcal{G}'$ and obtain $(\overline{\boldsymbol{x}}, \overline{\lambda})$, the following performance bound is guarantee for* (Approx-OPT)*:*

$$\widetilde{\mathcal{F}}(\overline{\boldsymbol{x}}) \geq \max_{\boldsymbol{x}}\{\widetilde{F}(\boldsymbol{x})\} - \frac{\epsilon'(\mathcal{H} + 2\mathcal{C})}{\sum_{s \in \mathcal{L}} \sum_{\tau \in \Delta(s)} \exp\left(\frac{U(\tau; U^x \boldsymbol{e})}{\mu}\right)} \tag{49}$$

where the constant $\mathcal{C} = \left(\kappa \max_s\{\rho^s\} + \overline{\lambda}\rho\right)$.

Proof. Let $(\mathbf{x}^*, \lambda^*)$ be the output of the bisection (aka binary search) to solve the restricted interdiction problem (Approx-OPT). Based on the above two lemmas, we now try to bound the gap $|\overline{\lambda} - \lambda^*|$ (based on which we can bound $\max_{\mathbf{x}\in\mathcal{X}} \widetilde{\mathcal{F}}(\mathbf{x}) - \widetilde{\mathcal{F}}(\overline{\mathbf{x}})$ as we will explain later).

First, since $(\overline{\mathbf{x}}, \overline{\lambda})$ is an output of binary search for solving (OPT) with the modified network $\mathcal{G}'$, we have $|g(\overline{\mathbf{x}}, \overline{\lambda}|\mathcal{G}')| \leq \xi$, where ξ is a positive constant depending on the precision of the binary search. In fact, this constant can be made arbitrarily small and therefore, we can remove it from the rest of our proof for the sake of presentation — that is, in the following, we simply consider $|g(\overline{\mathbf{x}}, \overline{\lambda}|\mathcal{G}')| = 0$. Now according to Lemma 3, we have:

$$|g(\overline{\mathbf{x}}, \overline{\lambda}|\mathcal{G}') - \widetilde{g}(\overline{\mathbf{x}}, \overline{\lambda})| \leq \epsilon' \left(\kappa \max_s\{\rho^s\} + \overline{\lambda}\rho\right) = \epsilon'\mathcal{C} \tag{50}$$

Since $|g(\overline{\mathbf{x}}, \overline{\lambda}|\mathcal{G}')| = 0$ we have $|\widetilde{g}(\overline{\mathbf{x}}, \overline{\lambda})| \leq \epsilon'\mathcal{C}$.

On the other hand, since $(\mathbf{x}^*, \lambda^*)$ is the result of binary search for (Approx-OPT), we have: $\widetilde{g}(\mathbf{x}^*, \lambda^*) = 0$ and $\lambda^* = \max_{\mathbf{x}\in\mathcal{X}} \widetilde{F}(\mathbf{x})$. We denote by:

$$\widetilde{K}(\lambda) = \max_{\mathbf{x}\in\mathcal{X}} \widetilde{g}(\mathbf{x}, \lambda)$$

which is monotonic decreasing in λ. In addition, $\widetilde{K}(\lambda^*) = \widetilde{g}(\mathbf{x}^*, \lambda^*) = 0$. According to Lemma 4:

$$\widetilde{K}(\overline{\lambda}) - \epsilon'\mathcal{H} \leq \widetilde{g}(\overline{\mathbf{x}}, \overline{\lambda}) \leq \widetilde{K}(\overline{\lambda}). \tag{51}$$

As a result, we obtain the following chain of inequalities:

$$\epsilon'\mathcal{H} \geq |\widetilde{K}(\overline{\lambda}) - \widetilde{g}(\overline{\mathbf{x}}, \overline{\lambda})| \overset{(a)}{\geq} |\widetilde{K}(\overline{\lambda})| - |\widetilde{g}(\overline{\mathbf{x}}, \overline{\lambda})| \geq |\widetilde{K}(\overline{\lambda})| - \epsilon'\mathcal{C} \tag{52}$$

$$\implies |\widetilde{K}(\overline{\lambda})| \leq \epsilon'(\mathcal{H} + \mathcal{C}) \tag{53}$$

where (a) is due to the triangle inequality. We further have $\widetilde{K}(\lambda^*) = 0$, leading to:

$$|\widetilde{K}(\overline{\lambda}) - \widetilde{K}(\lambda^*)| \leq \epsilon'(\mathcal{H} + \mathcal{C})$$

Since $\widetilde{K}(\lambda)$ is differentiable in λ, the mean value theorem tells us that there is $\alpha \in [\overline{\lambda}, \lambda^*]$ such that:

$$|\widetilde{K}(\overline{\lambda}) - \widetilde{K}(\lambda^*)| = \widetilde{K}'(\alpha)|\overline{\lambda} - \lambda^*| \leq \epsilon'(\mathcal{H} + \mathcal{C}). \tag{54}$$

We denote by $\widetilde{\mathbf{x}}$ the solution to $\widetilde{K}(\alpha) = \max_{\mathbf{x}\in\mathcal{X}}\{\widetilde{g}(\mathbf{x}, \alpha)\}$. We can compute the gradient, $\widetilde{K}'(\alpha)$, using the Danskin's theorem, as follows:

$$|\widetilde{K}'(\alpha)| = \left(\sum_{s\in\mathcal{L}}\sum_{\tau\in\Delta(s)} \exp\left(\frac{U(\tau;\widetilde{\mathbf{x}})}{\mu}\right)\right) \geq \sum_{s\in\mathcal{L}}\sum_{\tau\in\Delta(s)} \exp\left(\frac{U(\tau;U^x\mathbf{e})}{\mu}\right),$$

Together with (54) we obtain the bound:

$$0 \leq \lambda^* - \overline{\lambda} \leq \frac{\epsilon'(\mathcal{H} + \mathcal{C})}{\sum_{s\in\mathcal{L}}\sum_{\tau\in\Delta(s)} \exp\left(\frac{U(\tau;U^x\mathbf{e})}{\mu}\right)}. \tag{55}$$

Given the above bound on $\lambda^* - \overline{\lambda}$, we are now going to bound $\max_{\mathbf{x}\in\mathcal{X}} \widetilde{\mathcal{F}}(\mathbf{x}) - \widetilde{\mathcal{F}}(\overline{\mathbf{x}})$. From the inequality $|\widetilde{g}(\overline{\mathbf{x}}, \overline{\lambda})| \leq \epsilon'\mathcal{C}$ claimed above (Equation 50), we have:

$$\left|\sum_{s\in\mathcal{L}}\sum_{\tau\in\Delta(s)} r^l(s, \overline{x}_s) \exp\left(\frac{U(\tau;\overline{\mathbf{x}})}{\mu}\right) - \overline{\lambda}\left(\sum_{s\in\mathcal{L}}\sum_{\tau\in\Delta(s)} \exp\left(\frac{U(\tau;\overline{\mathbf{x}})}{\mu}\right)\right)\right| \leq \epsilon'\mathcal{C}$$

which implies

$$\left|\widetilde{\mathcal{F}}(\overline{\mathbf{x}}) - \overline{\lambda}\right| \leq \frac{\epsilon'\mathcal{C}}{\sum_{s\in\mathcal{L}}\sum_{\tau\in\Delta(s)} \exp\left(\frac{U(\tau;\overline{\mathbf{x}})}{\mu}\right)} \leq \frac{\epsilon'\mathcal{C}}{\sum_{s\in\mathcal{L}}\sum_{\tau\in\Delta(s)} \exp\left(\frac{U(\tau;U^x\mathbf{e})}{\mu}\right)}$$

$$\implies \widetilde{\mathcal{F}}(\overline{\mathbf{x}}) \geq \overline{\lambda} - \frac{\epsilon'\mathcal{C}}{\sum_{s\in\mathcal{L}}\sum_{\tau\in\Delta(s)} \exp\left(\frac{U(\tau;U^x\mathbf{e})}{\mu}\right)}$$

As a result, we obtain the following bounds:

$$\lambda^* = \max_{\mathbf{x}\in\mathcal{X}} \widetilde{\mathcal{F}}(\mathbf{x}) \geq \widetilde{\mathcal{F}}(\overline{\mathbf{x}}) \geq \overline{\lambda} - \frac{\epsilon'\mathcal{C}}{\sum_{s\in\mathcal{L}}\sum_{\tau\in\Delta(s)} \exp\left(\frac{U(\tau;U^x\mathbf{e})}{\mu}\right)}$$

$$\implies \max_{\mathbf{x}\in\mathcal{X}} \widetilde{\mathcal{F}}(\mathbf{x}) - \widetilde{\mathcal{F}}(\overline{\mathbf{x}}) \leq \lambda^* - \overline{\lambda} + \frac{\epsilon'\mathcal{C}}{\sum_{s\in\mathcal{L}}\sum_{\tau\in\Delta(s)} \exp\left(\frac{U(\tau;U^x\mathbf{e})}{\mu}\right)} \leq \frac{\epsilon'(\mathcal{H}+2\mathcal{C})}{\sum_{s\in\mathcal{L}}\sum_{\tau\in\Delta(s)} \exp\left(\frac{U(\tau;U^x\mathbf{e})}{\mu}\right)}.$$

which concludes our proof. □

We now get back to **the main proof of Theorem 4**. Essentially, Theorem 4 is a direct result of Theorem 2 and Lemma 5 with the constant:

$$\mathcal{U} = \frac{\mathcal{H}+2\mathcal{C}}{(\min_{\mathbf{x}\in\mathcal{X}} \widetilde{\mathcal{F}}(\mathbf{x}) + \kappa)\sum_{s\in\mathcal{L}}\sum_{\tau\in\Delta(s)} \exp\left({}^{U(\tau;U^x\mathbf{e})}/_{\mu}\right)}.$$

We complete the proof.

11 NP-harness

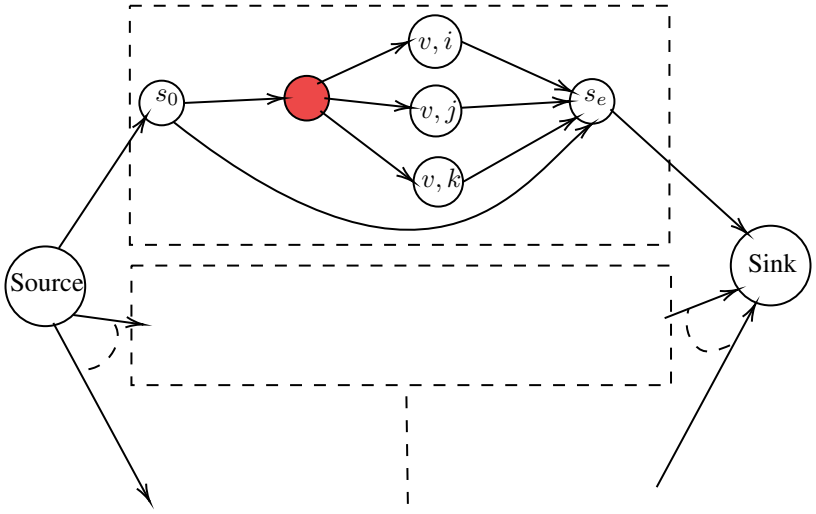
Our Theorem 5 below shows that the original problem (OPT) is essentially NP-Hard for a rational adversary.

Theorem 5. *The problem* (OPT) *is NP-Hard for* $\mu = 0$.

Proof. We do a reduction from exact 3-Cover problem, where given m items $\{1,\ldots,m\}$ and a collection of n subsets $\{S_1,\ldots,S_n\}$ with $S_v \subset \{1,\ldots,m\}$ each of size 3, i.e., $|S_v| = 3$ for $v \in \{1,\ldots,n\}$, the decision problem is whether there is a cover that contains each item exactly once. This is a known NP-Hard problem. Also, clearly any valid cover must be of $m/3$ number of subsets.

Game instance construction given the 3-Cover problem Given an exact 3-Cover problem, we form an instance of our network security game as follows: The critical nodes can be one of $m+1$ types, among which the first m types are labeled $1,\ldots,m$ and the last type is labeled *red*. In addition, there are a total of $n - m/3 + m$ defender resources. Among these resources, $n - m/3$ resources, denoted by $R_1,\ldots,R_{n-m/3}$, can defend nodes of type red (we call these the red resources). The remaining m resources are denoted by $r_1,\ldots,r_m$, where resource r_j can defend a node of type j.

We form n sub-graphs — each sub-graph corresponds to a subset $S_v = \{i,j,k\}$ shown below in the picture. The sub-graph for S_v has one critical node of type red and three critical nodes labeled $(v,i),(v,j),(v,k)$ of types i,j,k respectively. There is an initial non-critical node s_0 and an end non-critical node s_e in the sub-graph. A direct edge also connects s_0 to s_e, called a dummy edge. We can join all the n sub-graphs by making a source node and connect the source node to all s_0 for every sub-graph and a sink node and connect all s_e of each sub-graph to the sink node (see below).



For any node, p denotes the probability the defender protects that node. We set the payoff of the adversary for the critical nodes as $u_a(p) = -50p + 50(1-p) - Kp\log p = 50 - 100p - Kp\log p$ where the constant $K = \frac{100}{\log\left(\frac{n-m/3+1}{n-m/3+0.5}\right)} > 0$. The attacker payoff of skipping all critical nodes via the bottom edge in each sub-graph is 0. On the other hand, the defender's payoff, when an adversary visits a critical node is set to $u_d(p) = -100(1-p) - \epsilon$ for some small $\epsilon > 0$. Thus, the defender's payoff is always strictly negative if the attacker crosses any critical node. When the adversary does not visit any critical node, the defender payoff is 0. This means that any defender optimal strategy gives the defender a maximum payoff of at most 0.

Problem reduction. We claim that there exists an exact 3-Cover if and only if the defender's optimal expected payoff for the corresponding network interdiction game is 0.

First, assume there is an exact 3-Cover, then we show that the following strategy provides a payoff of 0 to the defender: (i) the defender allocates the $n - m/3$ red resources $R_1, \ldots, R_{n-m/3}$ to the red nodes in the $n - m/3$ sub-graphs corresponding to non-cover subsets; and (ii) for the $m/3$ sub-graphs corresponding to subsets in cover, the defender allocates the m resources $r_1, \ldots, r_m$ to the three nodes in each sub-graph. This ensures that either the red node or the three nodes $(v, i), (v, j), (v, k)$ are completely protected in every sub-graph S_v. Given this strategy of the defender, we note that in any non-dummy path, there are exactly two critical nodes — one node will be protected by the defender with a probability of 1 and the other critical node is protected with a probability of zero. As a result, if the adversary chooses this path, the adversary will obtain a total expected payoff over these two nodes as equal to $-50 + 50 = 0$ (given $p \log p = 0$ when p is either 0 or 1). Breaking ties in favor of defender [Leitmann, 1978], the adversary will choose one of the bottom dummy edges, providing an expected payoff of zero for the defender. As we discussed previously, the maximum payoff the defender can achieve is at most 0. Therefore, the above strategy is an optimal strategy for the defender that leads to the maximum defender payoff of 0.

Next, assume that the defender gets an optimal expected payoff of 0 and the equilibrium strategy is a vector of probability values for each critical node $\mathbf{p}^*$. As the expected payoff is 0, the adversary must have chosen one of the bottom edges. Let us analyze one path through the critical nodes that has $p^*_{v,r}$ on red node and $p^*_{v,i}$ on the other node of type i. The adversary payoff for choosing this path is $100 - 100(p^*_{v,r} + p^*_{v,i}) - Kp^*_{v,r} \log p^*_{v,r} - Kp^*_{v,i} \log p^*_{v,i}$. This adversary payoff for this path must be ≤ 0 (since adversary chooses the bottom edge). Or by rearranging and dividing by 100,

$$p^*_{v,r}(1 + (K/100) \log p^*_{v,r}) + p^*_{v,i}(1 + (K/100) \log p^*_{v,i}) \geq 1 \tag{56}$$

Based on Eq. 56, we are going to show that the red nodes are uncovered (i.e., $p^*_{v,r} = 0$) for exactly $m/3$ sub-graphs. First, since we have $n - m/3$ defender resources that can cover red nodes and we have n red nodes in total (i.e., one red node for each of n sub-graphs), the number of red nodes are uncovered (i.e., $p^*_{v,r} = 0$) is $\leq m/3$. Furthermore, we will show by contradiction that the other situation of $p^*_{v,r} = 0$ for $< m/3$ sub-graphs cannot occur. To prove by contradiction, assume that $p^*_{v,r} = 0$ for $< m/3$ sub-graphs or in other words $p^*_{v,r} > 0$ for $\geq n - m/3 + 1$ graphs. In the following, we prove that this assumption leads to a contradiction and hence this assumption cannot hold.

We observe that $p^*_{v,i}(1 + (K/100) \log p^*_{v,i}) \leq 1$. Thus, from Eq. 56, we obtain:

$$p^*_{v,r}(1 + (K/100) \log p^*_r) \geq 0 \tag{57}$$

Let $P \subset \{1, \ldots, n\}$ be the subset for which $p^*_{v,r} > 0$ for $v \in P$ (and by our assumption $|P| \geq n - m/3 + 1$). For any such $p^*_{v,r} > 0$ with $v \in P$, from Eq. 57, we must have $1 + (K/100) \log p^*_{v,r} \geq 0$, which when rearranged gives $p^*_{v,r} \geq \exp(-100/K)$. Summing over all $v \in P$, we get

$$\sum_{v \in P} p^*_{v,r} \geq |P| \exp(-100/K)$$

The LHS above is $\sum_{v \in P} p^*_{v,r} \leq n - m/3$ (as $p^*_{v,r} = 0$ for $v \notin P$ and there are only $n - m/3$ red resources). With $K = \frac{100}{\log\left(\frac{n-m/3+1}{n-m/3+0.5}\right)}$ and the fact that $|P| \geq n - m/3 + 1$, we get the RHS is $\geq n - m/3 + 0.5$. This is a contradiction. Thus, the assumption that $p^*_{v,r} = 0$ for $< m/3$ sub-graphs does not hold.

Therefore, the red nodes are uncovered (i.e., $p^*_{v,r} = 0$) for exactly $m/3$ sub-graphs. We note that if $p^*_{v,r} = 0$, then $p^*_{v,i} = 1$ in order to satisfy Eq. 56. And with the same reasoning for other non-dummy paths in the same sub-graph $S_v = \{i, j, k\}$, if $p^*_{v,r} = 0$ then $p^*_{v,i} = p^*_{v,j} = p^*_{v,k} = 1$. Remember that each defender resource r_j among the m (non-red) defender resources can only protect a node of type j. This means the m non-red nodes in these $m/3$ sub-graphs are from m different types. As a result, we obtain the satisfied cover for the original 3-Cover problem — each subset belonging to this cover has three items that correspond to the three non-red nodes of one of the above $m/3$ sub-graphs. □

12 Choosing N and K for the LiSD

We provide experiments to justify our choices of N and K for the **LiSP** described. We first fix $N = 90$ and vary K from 5 to 100. For each value of K, we generate 10 independent instances and solve them using the MILP approach. The means and standard deviations of the optimal values are reported in Tab. 3. It can be seen that the optimal values given by $K = 20$ are only 0.3% different from those given by the largest value of K (i.e., $K = 100$). This indicates that $K = 20$ would be sufficient for the PL approximation to achieve insignificant approximation errors. We further fix $K = 20$ and vary N from 20 to 150. Each column is also computed using 10 independent game instances of the same size and report optimal values in Tab. 4. It can be seen that the optimal values given by $N = 90$ are only about 3% different from those from the largest value of N, i.e., $N = 150$. We therefore select $N = 90$ for the MILP experiments.

We choose $N = 90$ and $K = 20$ for the **LiSD** approach. According to the above analyses, these choices would suffice to guarantee low practical approximation errors stemming from both path-sampling and PL approximation. We use GUROBI (a SOTA MILP solver) to solve (MILP). All our experiments were run on a 2.1 GHz CPU with 128GB RAM.

K	5	10	20	30	40	100
mean	3.13	3.15	3.16	3.17	3.17	3.17
std	0.11	0.12	0.12	0.12	0.12	0.12

Table 3: Optimal values given by **LiSD** with different values of K.

N	30	60	90	100	120	150
mean	3.10	3.15	3.22	3.18	3.15	3.20
std	0.18	0.18	0.09	0.10	0.12	0.11

Table 4: Optimal values given by **LiSD** with different values of N.

13 Zero-sum Game Model

13.1 Problem Formulation

In this section we discuss a zero-sum game model that is often used in adversarial settings, in which the aim of the defender is to minimize the expected utility of the adversary. The adversary's expected utility can be computed as follows:

$$\mathcal{E}^f(\mathbf{x}) = \sum_{\tau\in\Omega} U(\tau;\mathbf{x}) \frac{\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)}{\sum_{\tau'\in\Omega}\exp\left(\frac{U(\tau';\mathbf{x})}{\mu}\right)}$$

The zero-sum game model can then be formulated as follows:

$$\min_{\mathbf{x}\in\mathcal{X}} \quad \mathcal{E}^f(\mathbf{x}) \tag{OPT-zerosum}$$

which is generally non-convex in $\mathbf{x}$. Since it shares the same structure with the non-zero-sum game model considered in the main body of the paper, our approximation method based on the restricted problem still applies. Here, instead of directly solve the non-convex problem (OPT-zerosum), we propose to optimize the following log-sum objective, which is more tractable to handle

$$\Gamma(\mathbf{x}) = \mu \log\Big(\sum\nolimits_{\tau\in\Omega}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)\Big)$$

It can be seen that $\Gamma(\mathbf{x})$ has a log-sum-exp convex form of a geometric program, thus it is convex [Boyd *et al.*, 2004]. From the results in Section 6.1, we further see that $\Gamma(\mathbf{x})$ can be computed by solving a system of linear equations, which can be done in poly-time. Thus, the optimization problem $\max_{\mathbf{x}}\Gamma(\mathbf{x})$ can be solved in poly-time. We discuss in the following a connection between (OPT-zerosum), the alternative formulation $\max_{\mathbf{x}}\Gamma(\mathbf{x})$ and the a classical shortest-path network interdiction problem [Smith and Song, 2020; Israeli and Wood, 2002]. To facilitate explanation of this point, let us consider the following shortest-path network interdiction problem:

$$\min_{\mathbf{x}\in\mathcal{X}} \quad \Big\{T(\mathbf{x}) = \max_{\tau\in\Omega} \quad U(\tau;\mathbf{x})\Big\}. \tag{OPT-shortest-path}$$

It is known that the above shortest-path network interdiction problem can be formulated as a mixed-integer linear program and is NP-hard [Israeli and Wood, 2002]. We first bound the gap between $\Gamma(\mathbf{x})$ and $T(\mathbf{x})$ for any $\mathbf{x}\in\mathcal{X}$ in Lemma 6 below

Lemma 6. *For $\boldsymbol{x}\in\mathcal{X}$, let $\tau^* = \text{argmax}_{\tau\in\Omega}U(\tau;\boldsymbol{x})$ (i.e., the best trajectory which gives the highest adversary utility), $\Omega^* = \{\tau|\ U(\tau;\boldsymbol{x}) = U(\tau^*;\boldsymbol{x})\}$ (i.e., the set of all trajectories with the same highest utility), and $\alpha = \{U(\tau^*;\boldsymbol{x}) - \max_{\tau\in\Omega\setminus\Omega^*}U(\tau;\boldsymbol{x})\}$, then we have:*

$$|\Gamma(\boldsymbol{x}) - T(\boldsymbol{x})| \le \mu\log\left(|\Omega^*| + \frac{|\Omega\setminus\Omega^*|}{\exp(\alpha/\mu)}\right).$$

As a result, $\lim_{\mu\to 0}\Gamma(\boldsymbol{x}) = U(\tau^*)$.

Proof. We can write:

$$\begin{aligned}\Gamma(\mathbf{x}) &= \mu \log\left(\sum_{\tau\in\Omega}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)\right)\\ &= \mu \log\left(|\Omega^*|\exp\left(\frac{U(\tau^*;\mathbf{x})}{\mu}\right)+\sum_{\tau\in\Omega\setminus\Omega^*}\exp\left(\frac{U(\tau;\mathbf{x})}{\mu}\right)\right)\\ &\le \mu \log\left(|\Omega^*|\exp\left(\frac{U(\tau^*;\mathbf{x})}{\mu}\right)+(|\Omega\setminus\Omega^*|)\exp\left(\frac{U(\tau^*;\mathbf{x})-\alpha}{\mu}\right)\right)\\ &= U(\tau^*;\mathbf{x})+\mu\log\left(|\Omega^*|+(|\Omega\setminus\Omega^*|)\exp\left(\frac{-\alpha}{\mu}\right)\right) \qquad (58)\end{aligned}$$

Moreover, we have $\Gamma(\mathbf{x}) \ge U(\tau^*;\mathbf{x})$. Combine this with (58) we obtain the desired inequality. The limit $\lim_{\mu\to 0}\Gamma(\mathbf{x}) = U(\tau^*)$ is just a direct result of this equality, concluding our proof. □

Combine Lemma 6 with Proposition 1, we obtain a bound for $|\mathcal{E}^f(\mathbf{x}) - \Gamma(\mathbf{x})|$

Lemma 7. *Let $L^* = \max_{\tau\in\Omega,\boldsymbol{x}\in\mathcal{X}}|U(\tau;\boldsymbol{x})|$, we have*

$$|\mathcal{E}^f(\boldsymbol{x}) - \Gamma(\boldsymbol{x})| \le \frac{L^*+1}{1+\frac{|\Omega^*|}{|\Omega\setminus\Omega^*|}\exp\left(\frac{\alpha}{\mu}\right)} + \mu\log\left(|\Omega^*| + \frac{|\Omega\setminus\Omega^*|}{\exp\left(\frac{\alpha}{\mu}\right)}\right)$$

We are now ready to assess the quality of a solution given by the alternative formula $\max_{\mathbf{x}}\Gamma(\mathbf{x})$ and the zero-sum game ones (OPT-zerosum) and (OPT-shortest-path). Let $\Gamma^* = \max_{\mathbf{x}}\Gamma(\mathbf{x})$, $\mathcal{E}^*$, T^* be the optimal value of (OPT-zerosum), (OPT-shortest-path), and $\bar{\mathbf{x}}$ be the optimal solution to $\max_{\mathbf{x}}\Gamma(\mathbf{x})$. Given any $\mathbf{x}\in\mathcal{X}$, let:

$$\alpha(\mathbf{x}) = \max_{\tau\in\Omega}U(\tau;\mathbf{x}) - \max\{U(\tau;\mathbf{x})|\ \tau\in\Omega,\ U(\tau;\mathbf{x}) < \max_{\tau\in\Omega}U(\tau;\mathbf{x})\}$$

Intuitively, $\alpha(\mathbf{x})$ is the adversary loss in utility if the adversary chooses the second best trajectory instead of the optimal one. In addition, let $C(\mathbf{x})$ be the number of best paths in Ω, that is, $C(\mathbf{x}) = |\operatorname{argmax}_{\tau\in\Omega}U(\tau;\mathbf{x})|$. We have the following results bounding the gaps between the convex problem $\max_{\mathbf{x}}\Gamma(\mathbf{x})$ and two baselines, i.e., the classical shortest-path network interdiction and its bounded rational version, as functions of μ. The results imply that the optimal values and optimal solutions to $\max_{\mathbf{x}}\Gamma(\mathbf{x})$ converge to those of (OPT-shortest-path) and (OPT-zerosum) when μ goes to zero.

Proposition 6. *Let $\boldsymbol{x}^* = argmax_{\boldsymbol{x}}\Gamma(\boldsymbol{x})$ and*

$$\begin{aligned}\kappa_1(\mu) &= \max_{\boldsymbol{x}}\left\{\mu\log\left(|C(\boldsymbol{x}) + \frac{|\Omega|-|C(\boldsymbol{x})}{\exp\left(\frac{\alpha(\boldsymbol{x})}{\mu}\right)}\right)\right\}\\ \kappa_2(\mu) &= \kappa_1(\mu) + \max_{\boldsymbol{x}}\left\{\mu\frac{L^*+1}{1+\frac{|C(\boldsymbol{x})}{|\Omega|-|C(\boldsymbol{x})}\exp\left(\frac{\alpha(\boldsymbol{x})}{\mu}\right)}\right\} \qquad (59)\end{aligned}$$

The following results hold

(i) $|\Gamma^* - T^*| \le \kappa_1(\mu)$, *and* $|T(\boldsymbol{x}^*) - \max_{\boldsymbol{x}}\{T(\boldsymbol{x})\}| \le 2\kappa_1(\mu)$

(ii) $|\Gamma^* - \mathcal{E}^*| \le \kappa_2(\mu)$, *and* $|\mathcal{E}^f(\boldsymbol{x}^*) - \max_{\boldsymbol{x}}\{\mathcal{E}^f(\boldsymbol{x})\}| \le 2\kappa_2(\mu)$

(iii) $\lim_{\mu\to 0}\kappa_1(\mu) = \kappa_2(\mu) = 0$

Proof. For (i), we first note that $\Gamma(\mathbf{x}) \le T(\mathbf{x})$ for any $\mathbf{x}\in\mathcal{X}$. Thus $\Gamma^* \le T^*$. Let $\bar{\mathbf{x}}$ be an optimal solution to (OPT-shortest-path), we write:

$$\begin{aligned}|\Gamma^* - T^*| = T^* - \Gamma^* &= T(\bar{\mathbf{x}}) - \max_{\mathbf{x}}\Gamma(\mathbf{x})\\ &\le T(\bar{\mathbf{x}}) - \Gamma(\bar{\mathbf{x}})\\ &\overset{(a)}{\le} \kappa_1(\mu).\end{aligned}$$

Method	Number of nodes ($\lvert\mathcal{S}\rvert$) 20	40	60	80	100
Baseline	99.95 ± 0.00	99.88 ± 0.07	99.74 ± 0.18	99.69 ± 0.23	99.07 ± 0.52

Table 5: Objective values of the optimal solutions obtained from the Baseline as a percentage of the optimal objective value obtained using our approach for handling an exponential number of paths. We use $p = 0.8, \mu = 2$. 20 datasets were randomly generated for each setting and the mean and standard deviation are reported.

where (a) is due to Proposition 1. Moreover, considering the gap $|T(\mathbf{x}^*) - \max_{\mathbf{x}} T(\mathbf{x})|$, we have the chain of inequalities

$$\begin{aligned} |T(\mathbf{x}^*) - \max_{\mathbf{x}}\{T(\mathbf{x})\}| &\leq |T(\mathbf{x}^*) - \max_{\mathbf{x}}\{\Gamma(\mathbf{x})\}| + |\Gamma^* - T^*| \\ &= |T(\mathbf{x}^*) - \Gamma(\mathbf{x}^*)| + |\Gamma^* - T^*| \\ &\leq 2\kappa_1(\mu). \end{aligned}$$

For (ii), let $\widehat{\mathbf{x}}$ be an optimal solution to (OPT-zerosum). Similarly, we can write

$$\begin{aligned} |\Gamma^* - \mathcal{E}^*| = \mathcal{E}^* - \Gamma^* &= \mathcal{E}^f(\widehat{\mathbf{x}}) - \max_{\mathbf{x}} \Gamma(\mathbf{x}) \\ &\leq \mathcal{E}^f(\widehat{\mathbf{x}}) - \Gamma(\widehat{\mathbf{x}}) \\ &\overset{(b)}{\leq} \kappa_2(\mu). \end{aligned} \tag{60}$$

where (b) is due to Lemma 7. Moreover, considering the gap $|\mathcal{E}^f(\mathbf{x}^*) - \max_{\mathbf{x}} \mathcal{E}^f(\mathbf{x})|$, we write

$$\begin{aligned} |\mathcal{E}^f(\mathbf{x}^*) - \max_{\mathbf{x}}\{\mathcal{E}^f(\mathbf{x})\}| &\leq |\mathcal{E}^f(\mathbf{x}^*) - \max_{\mathbf{x}}\{\Gamma(\mathbf{x})\}| + |\Gamma^* - \mathcal{E}^*| \\ &= |\mathcal{E}^f(\mathbf{x}^*) - \Gamma(\mathbf{x}^*)| + |\Gamma^* - \mathcal{E}^*| \\ &\leq 2\kappa_2(\mu). \end{aligned}$$

The limits $\lim_{\mu \to 0} \kappa_1(\mu) = \kappa_2(\mu) = 0$ are obviously verified, which concludes the proof. □

In fact, we can control the adversary's rationality by adjusting μ, i.e., the adversary would be more rational as $\mu \to 0$ (perfectly rational if $\mu = 0$), and be irrational as $\mu \to \infty$.

13.2 Experiment Results for Zero-Sum Games

Note that since the log-sum alternative $\min_{\mathbf{x}}\{\Gamma(\mathbf{x})\}$ is a convex problem, both the Baseline and using gradient descent on top of our proposed approach to handle the exponential number of paths in Section 6 are able to solve it to optimality. However, we see in Table 5 the performance of the Baseline slightly tips off as the graph size $|\mathcal{S}|$ increases, due to the fact that the objective in the baseline is estimated by sampling and the number of paths blows up exponentially with $|\mathcal{S}|$.

As the rationality of the adversary increases (associated with the decrease in μ), we expect that the optimal defender reward will decrease as the adversary is able to take the best paths with a larger probability. Moreover, for a zero-sum game, by the guarantees in Proposition 6 we can claim that the optimal solution would converge to the solution of (OPT-shortest-path). We note both the reward decrease and convergence in Figure 1.

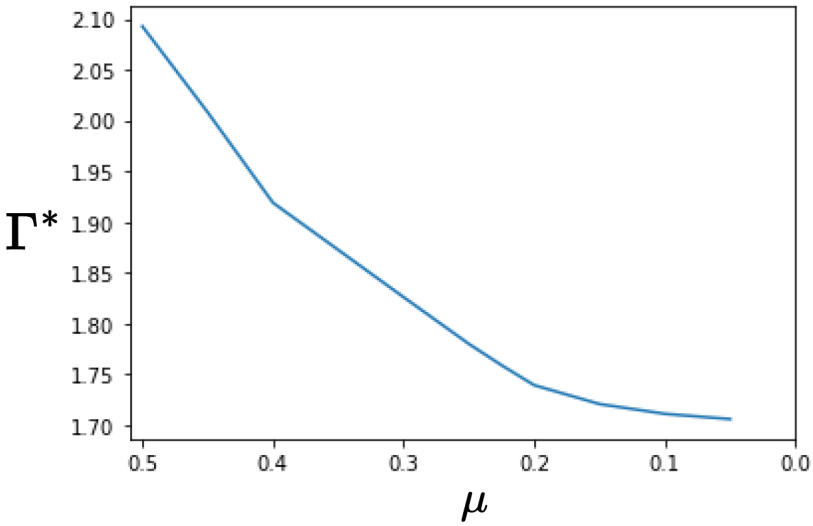


Figure 1: Optimal value Γ^* as a function of μ for a fixed synthetic dataset ($|\mathcal{S}| = 50$, p=0.8).